JAN 17

117

+

THE QUARTERLY JOURNAL OF MATHEMATICS

(OXFORD SERIES)

Volume 11 *No.* 43 *September* 1940

OEA

CONTENTS

OXFORD

AT THE CLARENDON PRESS

1940

Price 7s. 6d. net

PRINTED IN GREAT BRITAIN BY JOHN JOHNSON AT THE OXFORD UNIVERSITY PRESS

THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SERIES

Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, J. H. C. THOMPSON

With the co-operation of A. L. DIXON, W. L. FERRAR, G. H. HARDY, E. A. MILNE, E. C. TITCHMARSH

THE QUARTERLY JOURNAL OF MATHEMATICS (OXFORD SERIES) is published in March, June, September, and December, at a price of 7s. 6d. net for a single number with an annual subscription (for four numbers) of 27s. 6d. post free.

Papers, of a length normally not exceeding 20 printed pages of the Journal, are invited on subjects of Pure and Applied Mathematics, and should be addressed 'The Editors, Quarterly Journal of Mathematics, Clarendon Press, Oxford'. Contributions can be accepted in French and German, if in typescript (formulae excepted). Authors of papers printed in the Quarterly Journal will be entitled to 50 free offprints. Correspondence on the *subject-matter* of the Quarterly Journal should be addressed, as above, to 'The Editors', at the Clarendon Press. All other correspondence should be addressed to the Publisher (Humphrey Milford, Oxford University Press, Amen House, Warwick Square, London, E.C. 4).

HUMPHREY MILFORD
OXFORD UNIVERSITY PRESS
AMEN HOUSE, LONDON, E.C.4

ON A THEOREM DUE TO VINOGRADOW

By LOO-KENG HUA (*Kunming*)

[Received 26 September 1939]

1. Introduction

IT was proved by Vinogradow† that, *if P is a large positive integer, then*

$$\int_0^1 \cdots \int_0^1 \left| \sum_{x=1}^{P} e^{2\pi i(\alpha_k x^k + \ldots + \alpha_1 x)} \right|^r d\alpha_1 \ldots d\alpha_k = O(P^{r-\frac{1}{2}k(k+1)}),$$

for the greatest even integer r satisfying

$$r < Lk(k+1)(k+2)\log k,$$

where the value of L, which depends on k, is given by the following table:

k	2	3	4	5	6	7	8	9	10	11	12	13	$\geqslant 14$
L	4·81	4·45	4·34	4·30	4·29	4·23	4·22	4·18	4·16	4·15	4·14	4·12	4·10

The result is very important, and has numerous applications, e.g. to Waring's problem, to the estimation of trigonometrical sums as given by Vinogradow himself,‡ to Tarry's problem and to the simultaneous Waring's problem as given by the author.§ Now I am able to improve the result for small k: namely, *for the s given by the table*||

k	2	3	4	5	6	7	8	9	10
s	6	16	46	124	312	760	1778	4068	9190
r	80	293	721	1453	2582	4148	6318	9092	12644

we have

$$\int_0^1 \cdots \int_0^1 \left| \sum_{x=1}^{P} e^{2\pi i(\alpha_k x^k + \ldots + \alpha_1 x)} \right|^s d\alpha_1 \ldots d\alpha_k = O(P^{s-\frac{1}{2}k(k+1)+\epsilon}),$$

where the constant implied by the symbol O (and later by the equivalent symbol $\ll$) depends on k and ϵ only.

The method of proof is similar, in principle, to that used in my previous paper,†† but with a complicated modification.

We use $y\Delta f(x)$ to denote $f(x+y)-f(x)$; evidently $\Delta f(x)$ is a poly-

† *Recueil Math.* new series, **3** (1938), 435–71.

‡ Loc. cit., and *Travaux de l'Institut Math. de Tbilissi*, **5** (1938), 167–80.

§ Cf. the end of this paper.

|| The last line gives the values due to Vinogradow.

†† *Quart. J. of Math.* (Oxford), **9** (1938), 199–202.

nomial in x and y. The Δ operation will only be applied to the variable x. It is easy to verify that

$$y_1 \dots y_\mu \Delta^\mu x^\nu = 0 \quad (\mu > \nu),$$

$$y_1 \dots y_\mu \Delta^\mu x^\mu = \mu!\, y_1 \dots y_\mu,$$

and

$$y_1 \dots y_\mu \Delta^\mu x^{\mu+1} = y_1 \dots y_\mu w,$$

where w is a linear form in $y_1, \dots, y_\mu$ and x with integer coefficients.

LEMMA 1. *Let $g_1(x), \dots, g_s(x)$ be polynomials in x, and let*

$$g(x) = \alpha_1 g_1(x) + \dots + \alpha_s g_s(x)$$

be of degree k. Let

$$F = \sum_{x=1}^{P} e^{2\pi i g(x)}.$$

Then

$$F^{2^\mu} \leqslant P^{2^\mu - 1} + P^{2^\mu - \mu - 1} \sum_{y_1}^{P} \dots \sum_{y_\mu}^{P} \sum_{x_{\mu+1}}^{P} {}^{*}\, e^{2\pi i y_1 \dots y_\mu \Delta^\mu g(x_{\mu+1})}$$

for $\mu = 1, 2, \dots, k-1$, where $\sum_{y}^{P}$ denotes a sum† *with $O(P)$ terms, and where * denotes the conditions $y_1 \dots y_\mu \Delta^\mu g_r(x_{\mu+1}) \neq 0$, for all those values of r for which $g_r(x)$ is of degree greater than μ.*

The proof is the same as that given in my previous paper.‡

THEOREM $A(k)$. *Let*

$$f(x) = a_0 x^k + a_1 x^{k-1} + \dots,$$

where a_0 is an integer $\leqslant 1$ and a_1 is an integer $\leqslant P$. Let

$$S_k = \sum_{x=1}^{P} e^{2\pi i (\alpha_k f(x) + \alpha_{k-2} x^{k-2} + \dots + \alpha_1 x)}.$$

Then we have

$$\int_0^1 \dots \int_0^1 |S_k|^\lambda \, d\alpha_1 \dots d\alpha_{k-2}\, d\alpha_k = O(P^{\lambda - \frac{1}{2}(k^2 - k + 2) + \epsilon}),$$

where the value of λ is given by the table

k	3	4	5	6	7	8	9	10
λ	10	32	86	220	536	1272	2930	6628.

THEOREM $B(k)$. *Let*

$$C_k = \sum_{x=1}^{P} e^{2\pi i (\alpha_k x^k + \dots + \alpha_1 x)}.$$

Then we have

$$\int_0^1 \dots \int_0^1 |C_k|^s \, d\alpha_1 \dots d\alpha_k = O(P^{s - \frac{1}{2}k(k+1) + \epsilon}),$$

where s is as given previously.

† The conditions of summation for y_2 may depend on the value of y_1, and so on.

‡ Loc. cit. 201.

The proofs of both theorems depend on one another; more precisely, I shall prove the truth of theorem $A(k)$ by using results in the proofs of $A(l_1)$ and $B(l_2)$ for $l_1 \leqslant k-1$ and $l_2 \leqslant k-1$, and the truth of theorem $B(k)$ by using results in the proofs of $A(l_1)$ and $B(l_2)$ for $l_1 \leqslant k$ and $l_2 \leqslant k-1$. The methods used differ for different values of k. Certainly there exists a uniform method for the induction, but such a method gives a worse result.

2. Proof of the theorems

2.01. We may assume that $f(x) = x^k$ in $A(k)$, provided that we make a certain slight modification in the enunciation. In fact,

$$\int_0^1 \dots \int_0^1 |S_k|^{2\mu}\, d\alpha_1 \dots d\alpha_{k-2}\, d\alpha_k$$

is equal to the number of solutions of

$$\begin{gathered} f(x_1)+\dots+f(x_\mu) = f(y_1)+\dots+f(y_\mu), \\ x_1^{k-2}+\dots+x_\mu^{k-2} = y_1^{k-2}+\dots+y_\mu^{k-2}, \\ \dots\dots\dots\dots \\ x_1+\dots+x_\mu = y_1+\dots+y_\mu, \end{gathered}$$

subject to

$$1 \leqslant x_\nu \leqslant P, \quad 1 \leqslant y_\nu \leqslant P \qquad (\nu = 1, 2, \dots, \mu).$$

Multiplying the equations by $k^k a_0^{k-1}$, $k^{k-2}a_0^{k-2}, \dots, ka_0$ respectively, they become

$$\sum_{\nu=1}^{\mu} \{(ka_0 x_\nu)^k + ka_1(ka_0 x_\nu)^{k-1}+\dots\} = \sum_{\nu=1}^{\mu} \{(ka_0 y_\nu)^k + ka_1(ka_0 y_\nu)^{k-1}+\dots\},$$

$$\begin{gathered} \sum_{\nu=1}^{\mu} (ka_0 x_\nu)^{k-2} = \sum_{\nu=1}^{\mu} (ka_0 y_\nu)^{k-2}, \\ \dots\dots\dots\dots \\ \sum_{\nu=1}^{\mu} (ka_0 x_\nu) = \sum_{\nu=1}^{\mu} (ka_0 y_\nu). \end{gathered}$$

Put $x'_\nu = ka_0 x_\nu + a_1$, $y'_\nu = ka_0 y_\nu + a_1$; then we have the system of equations

$$\left.\begin{gathered} x_1'^k+\dots+x_\mu'^k = y_1'^k+\dots+y_\mu'^k \\ x_1'^{k-2}+\dots+x_\mu'^{k-2} = y_1'^{k-2}+\dots+y_\mu'^{k-2} \\ \dots\dots\dots\dots \\ x'_1+\dots+x'_\mu = y'_1+\dots+y'_\mu \end{gathered}\right\} \tag{1}$$

subject to the conditions

$$a_0 k \mid (x'_\nu - a_1), \tag{2}$$

$$a_1+1 \leqslant x'_\nu \leqslant a_1+P. \tag{3}$$

Certainly
$$\int_0^1 \dots \int_0^1 |S_k|^{2\mu}\, d\alpha_1 \dots d\alpha_k$$
does not exceed the number of solutions of (1) with the condition (3) only. Thus we need only discuss the case with $f(x) = x^k$, provided that in Theorems $A(k)$, $B(k)$ we allow the range of summation to be any interval of length P with both end-points $\ll P$. From now onwards we suppose them enunciated in this slightly more general form.

2.02. We shall now prove that
$$\int_0^1 \dots \int_0^1 |S_k|^{2k}\, d\alpha_1 \dots d\alpha_k \ll P^{k+\epsilon}. \tag{4}$$
For this we require the following

LEMMA 2. *Let*
$$s_i = x_1^i + \dots + x_k^i \quad (i = 1, 2, \dots, k).$$
Then the symmetrical function
$$f = (s_1 - x_1) \dots (s_1 - x_k)$$
of $x_1, \dots, x_k$ can be expressed as a function of $s_1, \dots, s_{k-2}$ and s_k only.

Proof. We write
$$f = s_1^k - s_1^{k-1}\sigma_1 + \dots + (-1)^k \sigma_k,$$
where σ_i is the ith elementary symmetrical function of $x_1, \dots, x_k$. By a well-known result on symmetrical functions, we have
$$f = (-1)^k \sigma_k + (-1)^{k-1} \sigma_{k-1} s_1 + f_1(s_1, \dots, s_{k-2}). \tag{5}$$
By Newton's formulae, we have
$$(-1)^k k \sigma_k = -s_k + \sigma_1 s_{k-1} + (-1)^k \sigma_{k-1} s_1 + f_2(s_1, \dots, s_{k-2}),$$
and
$$(-1)^{k-1}(k-1)\sigma_{k-1} = -s_{k-1} + f_3(s_1, \dots, s_{k-2}).$$
Consequently,
$$\begin{aligned} k\{(-1)^k \sigma_k + (-1)^{k-1} \sigma_{k-1} s_1\} &= -s_k + \sigma_1 s_{k-1} - s_1 s_{k-1} + f_4(s_1, \dots, s_{k-2}) \\ &= -s_k + f_4(s_1, \dots, s_{k-2}). \end{aligned}$$
Combining this with (5) we have the lemma.

2.03. *Proof of* (4). The integral on the left of (4) does not exceed the number of solutions of
$$\left.\begin{aligned} x_1^k + \dots + x_k^k &= y_1^k + \dots + y_k^k \\ x_1^{k-2} + \dots + x_k^{k-2} &= y_1^{k-2} + \dots + y_k^{k-2} \\ &\cdots\cdots \\ x_1 + \dots + x_k &= y_1 + \dots + y_k \end{aligned}\right\} \tag{6}$$

subject to $x_\nu = O(P)$, $y_\nu = O(P)$.

Let $$l = x_1+...+x_k = y_1+...+y_k.$$

By Lemma 2 the equations (6) imply that

$$(l-x_1)...(l-x_k) = (l-y_1)...(l-y_k).$$

For given $y_1,..., y_k$ with the condition

$$(l-y_1)...(l-y_k) \neq 0, \tag{7}$$

the number of sets of values for $x_1,..., x_k$ does not exceed

$$d^k\{(l-y_1)...(l-y_k)\} = O(P^\epsilon),$$

where $d(n)$ denotes the number of divisors of n.

If (7) does not hold, then one of the y's and also one of the x's is equal to l. Suppose $x_k = y_k = l$. There are $O(P)$ choices for x_k, and the equations (6) imply that

$$x_1^{k-2}+...+x_{k-1}^{k-2} = y_1^{k-2}+...+y_{k-1}^{k-2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x_1+...+x_{k-1} = y_1+...+y_{k-1} = 0.$$

There are $O(P^{k-1})$ choices for $x_{k-1}, y_2,..., y_{k-1}$, and these determine y_1 uniquely, and determine $x_1,..., x_{k-2}$ with not more than $(k-2)!$ $(= O(1))$ possibilities. Hence in this case also there are $O(P^{k+\epsilon})$ solutions for (6). This proves (4).

2.04. We next prove that

$$\int_0^1 ... \int_0^1 |C_k|^{2(k+1)}\, d\alpha_1...d\alpha_k \ll P^{k+1+\epsilon}. \tag{8}$$

For this we require the following

LEMMA 3. *The system of equations*

$$x_1^i+...+x_{k+1}^i = y_1^i+...+y_{k+1}^i \quad (i = 1, 2,..., k) \tag{9}$$

implies a relation of the form

$$(x_{k+1}-y_{k+1})g(y_1,..., y_k, x_k, y_{k+1}, x_{k+1}) = (x_k-y_k)h(y_1,..., y_k, x_k),$$

where g and h are homogeneous polynomials of degree $k-1$ in the variables indicated. The homogeneous polynomial of degree $k-1$ in x_{k+1} and y_{k+1} only, contained in g, is not divisible by $x_{k+1}-y_{k+1}$, and the coefficients of x_k^{k-1} in h and x_{k+1}^{k-1} in g are constants different from zero.

Proof. Let

$$s_i = x_1^i+...+x_{k-1}^i, \qquad t_i = y_1^i+...+y_{k-1}^i.$$

We can write (9) in the form

$$s_\nu = t_\nu-(x_k^\nu-y_k^\nu)-(x_{k+1}^\nu-y_{k+1}^\nu) \quad (\nu = 1,..., k). \tag{10}$$

It is well known that

$$s_k-\sigma_1 s_{k-1}+\sigma_2 s_{k-2}+...+(-1)^{k-1}\sigma_{k-1} s_1 = 0, \tag{11}$$

where σ_i is the ith elementary symmetrical function of $x_1,..., x_{k-1}$. Since σ_i is expressible as a polynomial in $s_1,..., s_i$, we can write (11) in the form

$$s_k-s_1 s_{k-1}+\sigma_2(s_1, s_2)s_{k-2}+...+(-1)^{k-1}\sigma_{k-1}(s_1,..., s_{k-1})s_1 = 0. \tag{12}$$

Similarly,

$$t_k-t_1 t_{k-1}+\sigma_2(t_1, t_2)t_{k-2}+...+(-1)^{k-1}\sigma_{k-1}(t_1,..., t_{k-1})t_1 = 0. \tag{13}$$

If we substitute from (10) in (12), we obtain the left-hand side of (13), plus a number of terms, each of which is a product of powers of the t's and of factors of the type $x_k^\nu-y_k^\nu$, $x_{k+1}^\nu-y_{k+1}^\nu$. When we subtract from this the equation (13), and take over to the right-hand side all terms which do not contain any factor of the type $x_{k+1}^\nu-y_{k+1}^\nu$, we obtain a relation which is of the required form.

The part of g containing only x_{k+1} and y_{k+1} arises from those terms which contain only factors of the type $x_{k+1}^\nu-y_{k+1}^\nu$. All such terms contain at least two factors, except that term $-(x_{k+1}^k-y_{k+1}^k)$ which comes from s_k. Now this term is not divisible by $(x_{k+1}-y_{k+1})^2$, whereas all others are. This proves the assertion about g.

The coefficient of x_{k+1}^k in the expression (12) after substituting from (10) can be found by taking $s_\nu = -x_{k+1}^\nu$. This makes $\sigma_\nu = (-1)^\nu x_{k+1}^\nu$, and so the coefficient is

$$-1-(-1)(-1)+(1)(-1)-...+(-1)^{k-1}(-1)^{k-1}(-1) = -k. \tag{14}$$

Thus the coefficient of x_{k+1}^{k-1} in g is $-k$. Similarly for the coefficient of x_k^{k-1} in h.

2.05. *Proof of* (8). The integral on the left is the number of solutions of the equations (9) subject to $x_\nu = O(P)$, $y_\nu = O(P)$. By Lemma 3 the equations imply

$$(x_{k+1}-y_{k+1})g(y_1,..., y_k, x_k, y_{k+1}, x_{k+1}) = (x_k-y_k)h(y_1,..., y_k, x_k). \tag{15}$$

For given $y_1,\ldots, y_k, x_k$ satisfying

$$(x_k-y_k)h(y_1,\ldots,y_k,x_k) \neq 0, \tag{16}$$

the number of sets of x_{k+1}, y_{k+1} does not exceed the divisor function

$$d\{(x_k-y_k)h(y_1,\ldots,y_k,x_k)\} = O(P^\epsilon).$$

This follows from the fact that, by the second assertion of Lemma 3, the number of solutions of

$$x_{k+1}-y_{k+1} = c, \qquad g(y_1,\ldots,y_k,x_k,y_{k+1},x_{k+1}) = d$$

(c, d being non-zero integers) does not exceed the degree $k-1$ of g.

Further, if $\qquad (x_k-y_k)h(y_1,\ldots,y_k,x_k) = 0,$

then $\qquad (x_{k+1}-y_{k+1})g(y_1,\ldots,y_k,x_k,y_{k+1},x_{k+1}) = 0.$

In virtue of the last assertion of Lemma 3, these equations imply that, for given $y_1,\ldots, y_k$ and y_{k+1}, there are only $O(1)$ possible values for x_k and for x_{k+1}.

In either case, when $y_1,\ldots, y_k, x_k, x_{k+1}, y_{k+1}$ are known, there are only $O(1)$ possible sets of values for $x_1,\ldots, x_{k-1}$, by (9). Hence the number of solutions of (9) subject to $x_\nu \leqslant P$, $y_\nu \leqslant P$, is $O(P^{k+1+\epsilon})$. This proves (8).

2.06. $B(2)$ is the particular case $k = 2$ of (8).

2.07. *Proof of* $A(3)$. By Lemma 1 we have

$$|S_3|^4 \leqslant P^3+P\sum_{y_1}^{P}\sum_{y_2}^{P}\sum_{x_3}^{P}{}^{*}\, e^{2\pi i y_1 y_2 \Delta^2(\alpha_3 x_3^3+\alpha_1 x_3)},$$

where the * means that the variables are subject to $y_1 y_2 \Delta^2 x_3^3 \neq 0$. Multiplying this inequality throughout by $|S_3|^6$, and integrating with respect to α_1 and α_3, we have

$$\int_0^1\int_0^1 |S_3|^{10}\, d\alpha_1\, d\alpha_3 \leqslant P^3 \int_0^1\int_0^1 |S_3|^6\, d\alpha_1\, d\alpha_3 + PR,$$

where R denotes the number of solutions of

$$y_1 y_2 \Delta^2(x_3^3) = z_1^3+\ldots+z_3^3-z_4^3-\ldots-z_6^3,$$
$$y_1 y_2 \Delta^2(x_3^3) \neq 0,$$
$$0 = z_1+\ldots+z_3-z_4-\ldots-z_6,$$

with $z_\nu = O(P)$. Since $z_1,\ldots, z_5$ determine the other variables with only $O(P^\epsilon)$ possibilities, we have $R = O(P^{5+\epsilon})$. Thus, using (4) with $k = 3$, we have

$$\int_0^1\int_0^1 |S_3|^{10}\, d\alpha_1\, d\alpha_3 \leqslant P^{6+\epsilon}. \tag{17}$$

2.08. *Proof of* $B(3)$. By Lemma 1

$$|C_3|^4 \ll P^3+P\sum_{y_1}^{P}\sum_{y_2}^{P}\sum_{x_3}^{P}{}^{*}\, e^{2\pi i y_1 y_2 \Delta^2(\alpha_3 x_3^3+\alpha_2 x_3^2)}, \tag{18}$$

where the * means that the variables are subject to

$$y_1 y_2 \Delta^2(x_3^3) \neq 0, \qquad y_1 y_2 \Delta^2(x_3^2) \neq 0.$$

Multiplying this inequality throughout by $|C_3|^8$, and integrating with respect to α_1, α_2, α_3, using (8) with $k = 3$, we have

$$\int_0^1\int_0^1\int_0^1 |C_3|^{12}\, d\alpha_1\, d\alpha_2\, d\alpha_3 \ll P^{7+\epsilon}+PR,$$

where R is the number of solutions of

$$\begin{aligned} y_1 y_2 w &= z_1^3+...+z_4^3-z_5^3-...-z_8^3 \quad (y_1 y_2 w \neq 0),\\ 2y_1 y_2 &= z_1^2+...+z_4^2-z_5^2-...-z_8^2,\\ 0 &= z_1+...+z_4-z_5-...-z_8, \end{aligned}$$

where $w = \Delta^2(x_3^3) \ll P$ and $z_\nu \ll P$.

For any fixed w the number of solutions of

$$(2z_1^3-wz_1^2)+...+(2z_4^3-wz_4^2) = (2z_5^3-wz_5^2)+...+(2z_8^3-wz_8^2),$$
$$z_1+...+z_4 = z_5+...+z_8$$

is $O(P^{5+\epsilon})$, by (4) with $k = 3$ and $f(z) = 2z^3-wz^2$. Therefore, $R = O(P^{6+\epsilon})$, and so

$$\int_0^1\int_0^1\int_0^1 |C_3|^{12}\, d\alpha_1\, d\alpha_2\, d\alpha_3 \ll P^{7+\epsilon}. \tag{19}$$

In the same way, multiplying (18) by $|C_3|^{12}$ and using (19) and

$$\int_0^1\int_0^1 |S_3|^{12}\, d\alpha_1\, d\alpha_3 \ll P^{8+\epsilon},$$

which is a trivial consequence of (17), we obtain

$$\int_0^1\int_0^1\int_0^1 |C_3|^{16}\, d\alpha_1\, d\alpha_2\, d\alpha_3 \ll P^{10+\epsilon}.$$

This is $B(3)$.

2.09. *Proof of* $A(4)$. By Lemma 1 we have

$$|S_4|^8 \ll P^7+P^4\sum_{y_1}^{P}\sum_{y_2}^{P}\sum_{y_3}^{P}\sum_{x_4}^{P}{}^{*}\, e^{2\pi i y_1 y_2 y_3 \Delta^3(\alpha_4 x_4^4+\alpha_2 x_4^2+\alpha_1 x_4)},$$

where the * denotes that $y_1 y_2 y_3 \Delta^3 x_4^4 \neq 0$. Multiplying by $|S_4|^8$ and integrating, and using (4) with $k = 4$, we have

$$\int_0^1\int_0^1\int_0^1 |S_4|^{16}\, d\alpha_1\, d\alpha_2\, d\alpha_4 \ll P^{11+\epsilon}+P^4R,$$

where R is the number of solutions of

$$y_1 y_2 y_3 \Delta^3 x_4^4 = z_1^4+...+z_4^4-z_5^4-...-z_8^4 \quad (y_1 y_2 y_3 \Delta^3 x_4^4 \neq 0),$$

$$0 = z_1^2+...+z_4^2-z_5^2-...-z_8^2,$$

$$0 = z_1+...+z_4-z_5-...-z_8,$$

with $z_\nu \ll P$. Using

$$\int_0^1\int_0^1 |C_2|^8\, d\alpha_1\, d\alpha_2 \ll P^{5+\epsilon},$$

which is a trivial consequence of $B(2)$, we have $R \ll P^{5+\epsilon}$. Thus

$$\int_0^1\int_0^1\int_0^1 |S_4|^{16}\, d\alpha_1\, d\alpha_2\, d\alpha_4 \ll P^{11+\epsilon}. \tag{20}$$

Repeating the argument, but multiplying by $|S_4|^{16}$ and $|S_4|^{24}$, we obtain successively

$$\int_0^1\int_0^1\int_0^1 |S_4|^{24}\, d\alpha_1\, d\alpha_2\, d\alpha_4 \ll P^{18+\epsilon}, \tag{21}$$

$$\int_0^1\int_0^1\int_0^1 |S_4|^{32}\, d\alpha_1\, d\alpha_2\, d\alpha_4 \ll P^{25+\epsilon}. \tag{22}$$

This is $A(4)$.

2.10. *Proof of* $B(4)$. By Lemma 1 we have

$$|C_4|^4 \ll P^3+P \sum_{y_1}^{P}\sum_{y_2}^{P}\sum_{x_3}^{P}{}^{*}\, e^{2\pi i y_1 y_2 \Delta^2(\alpha_4 x_3^4+\alpha_3 x_3^3+...)},$$

where the * denotes that the summation is subject to

$$y_1 y_2 \Delta^2 x_3^4 \neq 0, \qquad y_1 y_2 \Delta^2 x_3^3 \neq 0.$$

Multiplying the inequality by $|C_4|^{10}$ and integrating, we have, using (8),

$$\int_0^1\int_0^1\int_0^1\int_0^1 |C_4|^{14}\, d\alpha_1\, d\alpha_2\, d\alpha_3\, d\alpha_4 \ll P^{8+\epsilon}+PR,$$

where R denotes the number of solutions of

$$y_1 y_2 \Delta^2 x_3^4 = z_1^4 + \dots - z_{10}^4 \quad (y_1 y_2 \Delta^2 x_3^4 \neq 0),$$
$$y_1 y_2 w = z_1^3 + \dots - z_{10}^3 \quad (y_1 y_2 w \neq 0),$$
$$2y_1 y_2 = z_1^2 + \dots - z_{10}^2,$$
$$0 = z_1 + \dots - z_{10}.$$

(The usual conditions on the magnitude of w and the z's are to be understood.) For fixed w the number of solutions of

$$0 = (2z_1^3 - wz_1^2) + \dots - (2z_{10}^3 - wz_{10}^2),$$
$$0 = z_1 + \dots - z_{10}$$

is $\ll P^{6+\epsilon}$ by (17). Consequently, $R \ll P^{7+\epsilon}$, and

$$\int_0^1\int_0^1\int_0^1\int_0^1 |C_4|^{14}\, d\alpha_1\, d\alpha_2\, d\alpha_3\, d\alpha_4 \ll P^{8+\epsilon}. \tag{23}$$

By Lemma 1 we have

$$|C_4|^8 \ll P^7 + P^4 \sum_{y_1}^{P} \sum_{y_2}^{P} \sum_{y_3}^{P} \sum_{x_4}^{P}{}^{*} e^{2\pi i y_1 y_2 y_3 \Delta^3(\alpha_4 x_4^4 + \dots)},$$

where the * denotes that $y_1 y_2 y_3 \Delta^3 x_4^4 \neq 0$. Multiplying by $|C_4|^{14}$ and integrating, we have, by (23),

$$\int_0^1\int_0^1\int_0^1\int_0^1 |C_4|^{22}\, d\alpha_1 \dots d\alpha_4 \ll P^{15+\epsilon} + P^4 R,$$

where R denotes the number of solutions of

$$y_1 y_2 y_3 w = z_1^4 + \dots - z_{14}^4 \quad (y_1 y_2 y_3 w \neq 0),$$
$$6y_1 y_2 y_3 = z_1^3 + \dots - z_{14}^3,$$
$$0 = z_1^2 + \dots - z_{14}^2,$$
$$0 = z_1 + \dots - z_{14}.$$

Clearly R does not exceed $P^{1+\epsilon}$ times the number of solutions of

$$(6z_1^4 - wz_1^3) + \dots - (6z_{14}^4 - wz_{14}^3) = 0,$$
$$z_1^2 + \dots - z_{14}^2 = 0,$$
$$z_1 + \dots - z_{14} = 0$$

for a fixed w. By a trivial consequence of (4) we have

$$\int_0^1\int_0^1\int_0^1 |S_4|^{14}\, d\alpha_1\, d\alpha_2\, d\alpha_4 \ll P^{10+\epsilon},$$

and hence $R \ll P^{11+\epsilon}$. Thus,

$$\int_0^1\int_0^1\int_0^1\int_0^1 |C_4|^{22}\, d\alpha_1 ... d\alpha_4 \ll P^{15+\epsilon}. \tag{24}$$

Using the same method, but appealing to (20), (21), (22) instead of to (4), we obtain successively

$$\int_0^1\int_0^1\int_0^1\int_0^1 |C_4|^{30}\, d\alpha_1 ... d\alpha_4 \ll P^{22+\epsilon}, \tag{25}$$

$$\int_0^1\int_0^1\int_0^1\int_0^1 |C_4|^{38}\, d\alpha_1 ... d\alpha_4 \ll P^{29+\epsilon}, \tag{26}$$

$$\int_0^1\int_0^1\int_0^1\int_0^1 |C_4|^{46}\, d\alpha_1 ... d\alpha_4 \ll P^{36+\epsilon}. \tag{27}$$

This last is $B(4)$.

From now onwards we use the abbreviation

$$\int f\, d\alpha \quad \text{for} \quad \int_0^1 ... \int_0^1 f(\alpha_1,..., \alpha_n)\, d\alpha_1 ... d\alpha_n.$$

2.11. *Proof of* $A(5)$. By Lemma 1 we have

$$|S_5|^4 \ll P^3 + P \sum_{y_1}^{P} \sum_{y_2}^{P} \sum_{x_3}^{P}{}^{*}\, e^{2\pi i y_1 y_2 \Delta^2 g(x_3)},$$

where $g(x) = \alpha_5 x^5 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x$. Multiplying by $|S_5|^{10}$ and integrating, we have, by (4),

$$\int |S_5|^{14}\, d\alpha \ll P^{8+\epsilon} + PR,$$

where R denotes the number of solutions of

$$\begin{aligned} y_1 y_2 \Delta^2 x_3^5 &= z_1^5 + ... - z_{10}^5 \quad (y_1 y_2 \Delta^2 x_3^5 \neq 0),\\ y_1 y_2 w &= z_1^3 + ... - z_{10}^3 \quad (y_1 y_2 w \neq 0),\\ 2y_1 y_2 &= z_1^2 + ... - z_{10}^2,\\ 0 &= z_1 + ... - z_{10}. \end{aligned}$$

For a fixed w the number of solutions of

$$\begin{aligned} (2z_1^3 - wz_1^2) + ... - (2z_{10}^3 - wz_{10}^2) &= 0,\\ z_1 + ... - z_{10} &= 0 \end{aligned}$$

is $\ll P^{6+\epsilon}$, by $A(3)$. Hence $R \ll P^{7+\epsilon}$, and

$$\int |S_5|^{14}\, d\alpha \ll P^{8+\epsilon}. \tag{28}$$

By Lemma 1 we have

$$|S_5|^8 \ll P^7+P^4 \sum_{y_1}^{P}\sum_{y_2}^{P}\sum_{y_3}^{P}\sum_{x_4}^{P}{}^{*}\, e^{2\pi i y_1 y_2 y_3 \Delta^3 g(x_4)}.$$

Multiplying by $|S_5|^{14}$ and integrating, we have, by (28),

$$\int |S_5|^{22}\, d\alpha \ll P^{15+\epsilon}+P^4R,$$

where R denotes the number of solutions of

$$\begin{aligned} y_1 y_2 y_3 \Delta^3 x_3^5 &= z_1^5+\ldots-z_{14}^5 \quad (y_1 y_2 y_3 \Delta^3 x_3^5 \neq 0),\\ 6y_1 y_2 y_3 &= z_1^3+\ldots-z_{14}^3,\\ 0 &= z_1^2+\ldots-z_{14}^2,\\ 0 &= z_1+\ldots-z_{14}. \end{aligned}$$

Clearly,

$$R \ll P^{\epsilon} \int |C_2|^{14}\, d\alpha \ll P^{11+\epsilon}$$

by a trivial consequence of $B(2)$. Hence

$$\int |S_5|^{22}\, d\alpha \ll P^{15+\epsilon}. \tag{29}$$

By Lemma 1 we have

$$|S_5|^{16} \ll P^{15}+P^{11} \sum_{y_1}^{P}\sum_{y_2}^{P}\sum_{y_3}^{P}\sum_{y_4}^{P}\sum_{x_5}^{P}{}^{*}\, e^{2\pi i y_1 y_2 y_3 y_4 \Delta^4 g(x_5)}.$$

Multiplying by $|S_5|^{22}$ and integrating, we obtain

$$\int |S_5|^{38}\, d\alpha \ll P^{30+\epsilon}+P^{11}R,$$

where it is easily seen that

$$R \ll P^{\epsilon} \int |C_3|^{22}\, d\alpha \ll P^{16+\epsilon}$$

by a trivial consequence of $B(3)$. Thus

$$\int |S_5|^{38}\, d\alpha \ll P^{30+\epsilon}.$$

Repeating this process we obtain

$$\int |S_5|^{38+16\lambda}\, d\alpha \ll P^{30+15\lambda+\epsilon} \quad (\lambda = 0, 1, 2, 3). \tag{30}$$

The case $\lambda = 3$ is $A(5)$.

2.12. *Proof of* $B(5)$. By Lemma 1 we have

$$|C_5|^8 \ll P^7+P^4 \sum_{y_1}^{P}\sum_{y_2}^{P}\sum_{y_3}^{P}\sum_{x_4}^{P}{}^{*}\, e^{2\pi i y_1 y_2 y_3 \Delta^3(\alpha_5 x_4^5+\alpha_4 x_4^4+\ldots)}.$$

Multiplying this by $|C_5|^{12}$ and integrating, we have, by (8),

$$\int |C_5|^{20}\, d\alpha \ll P^{13+\epsilon}+P^4R;$$

and it is easily seen that

$$R \ll P^{1+\epsilon} \int |S_4|^{12}\, d\alpha \ll P^{9+\epsilon}$$

by a trivial consequence of (4). Hence

$$\int |C_5|^{20}\, d\alpha \ll P^{13+\epsilon}.$$

Repeating the argument, but appealing to (20), (21), (22) instead of to (4), we obtain

$$\int |C_5|^{20+8\lambda}\, d\alpha \ll P^{13+7\lambda+\epsilon} \quad (\lambda = 1, 2, 3). \tag{31}$$

Using Lemma 1 with $\mu = 4$, and appealing to (30), we obtain

$$\int |C_5|^{44+16\lambda}\, d\alpha \ll P^{34+15\lambda+\epsilon} \quad (\lambda = 1, 2, 3, 4, 5). \tag{32}$$

The case $\lambda = 5$ is $B(5)$.

2.13. *Proof of $A(6)$.* Using Lemma 1 with $\mu = 3$, and the results in the proof of $A(4)$, we obtain

$$\int |S_6|^{12+8\lambda}\, d\alpha \ll P^{6+7\lambda+\epsilon} \quad (\lambda = 1, 2, 3, 4). \tag{33}$$

Using Lemma 1 with $\mu = 4$, and $B(3)$, we obtain

$$\int |S_6|^{60}\, d\alpha \ll P^{49+\epsilon}. \tag{34}$$

Using Lemma 1 with $\mu = 5$, and the results in the proof of $B(4)$, we obtain

$$\int |S_6|^{60+32\lambda}\, d\alpha \ll P^{49+31\lambda+\epsilon} \quad (\lambda = 1, 2, 3, 4, 5). \tag{35}$$

The case $\lambda = 5$ is $A(6)$.

2.14. *Proof of $B(6)$.* Here we depart from the procedure of starting with (8), but start instead with

$$\int |C_6|^{16}\, d\alpha \ll P^{9+\epsilon},$$

which is an evident consequence of (8). Using Lemma 1 with $\mu = 3$, and the results in the proof of $A(4)$, we obtain

$$\int |C_6|^{16+8\lambda}\, d\alpha \ll P^{9+7\lambda+\epsilon} \quad (\lambda = 1, 2, 3). \tag{36}$$

By Lemma 1 with $\mu = 4$, and the results in the proof of $A(5)$,

$$\int |C_6|^{40+16\lambda}\, d\alpha \ll P^{30+15\lambda+\epsilon} \quad (\lambda = 1, 2, 3, 4, 5). \tag{37}$$

By Lemma 1 with $\mu = 5$, and the results in the proof of $A(6)$,

$$\int |C_6|^{120+32\lambda}\, d\alpha \ll P^{105+31\lambda+\epsilon} \quad (\lambda = 1, 2, 3, 4, 5, 6). \tag{38}$$

2.15. *Proof of* $A(7)$. As an evident consequence of (4), we have

$$\int |S_7|^{16}\,d\alpha \ll P^{9+\epsilon}.$$

By Lemma 1 with $\mu = 3$, and the results in the proof of $A(4)$, we have

$$\int |S_7|^{16+8\lambda}\,d\alpha \ll P^{9+7\lambda+\epsilon} \quad (\lambda = 1, 2, 3). \tag{39}$$

By Lemma 1 with $\mu = 4$, and the results in the proof of $A(5)$,

$$\int |S_7|^{40+16\lambda}\,d\alpha \ll P^{30+15\lambda+\epsilon} \quad (\lambda = 1, 2, 3, 4, 5). \tag{40}$$

By Lemma 1 with $\mu = 5$, and (27),

$$\int |S_7|^{152}\,d\alpha \ll P^{136+\epsilon}. \tag{41}$$

By Lemma 1 with $\mu = 6$, and the results in the proof of $B(5)$,

$$\int |S_7|^{152+64\lambda}\,d\alpha \ll P^{136+63\lambda+\epsilon} \quad (\lambda = 1, 2, 3, 4, 5, 6). \tag{42}$$

2.16. *Proof of* $B(7)$. As an evident consequence of (8), we have

$$\int |C_7|^{24}\,d\alpha \ll P^{16+\epsilon}.$$

As in the proof of $A(7)$, we obtain

$$\int |C_7|^{24+8\lambda}\,d\alpha \ll P^{16+7\lambda+\epsilon} \quad (\lambda = 1, 2), \tag{43}$$

$$\int |C_7|^{40+16\lambda}\,d\alpha \ll P^{30+15\lambda+\epsilon} \quad (\lambda = 1, 2, 3, 4, 5), \tag{44}$$

$$\int |C_7|^{120+32\lambda}\,d\alpha \ll P^{105+31\lambda+\epsilon} \quad (\lambda = 1, 2, 3, 4, 5, 6), \tag{45}$$

$$\int |C_7|^{312+64\lambda}\,d\alpha \ll P^{291+63\lambda+\epsilon} \quad (\lambda = 1, 2, 3, 4, 5, 6, 7). \tag{46}$$

2.17. *Proof of* $A(8)$. As an evident consequence of (4), we have

$$\int |S_8|^{24}\,d\alpha \ll P^{16+\epsilon}.$$

We have then

$$\int |S_8|^{24+8\lambda}\,d\alpha \ll P^{16+7\lambda+\epsilon} \quad (\lambda = 1, 2), \tag{47}$$

$$\int |S_8|^{40+16\lambda}\,d\alpha \ll P^{30+15\lambda+\epsilon} \quad (\lambda = 1, 2, 3, 4, 5), \tag{48}$$

$$\int |S_8|^{120+32\lambda}\,d\alpha \ll P^{105+31\lambda+\epsilon} \quad (\lambda = 1, 2, 3, 4, 5, 6), \tag{49}$$

$$\int |S_8|^{376}\,d\alpha \ll P^{354+\epsilon}, \tag{50}$$

$$\int |S_8|^{376+128\lambda}\,d\alpha \ll P^{354+127\lambda+\epsilon} \quad (\lambda = 1, 2, \ldots, 7). \tag{51}$$

2.18. *Proof of* $B(8)$. We have

$$\int |C_8|^{18+16\lambda}\, d\alpha \ll P^{9+15\lambda+\epsilon} \qquad (\lambda = 1, 2, 3, 4, 5, 6), \tag{52}$$

$$\int |C_8|^{114+32\lambda}\, d\alpha \ll P^{99+31\lambda+\epsilon} \qquad (\lambda = 1, 2, 3, 4, 5, 6), \tag{53}$$

$$\int |C_8|^{306+64\lambda}\, d\alpha \ll P^{285+63\lambda+\epsilon} \qquad (\lambda = 1, 2, \ldots, 7), \tag{54}$$

$$\int |C_8|^{754+128\lambda}\, d\alpha \ll P^{726+127\lambda+\epsilon} \qquad (\lambda = 1, 2, \ldots, 8). \tag{55}$$

2.19. *Proof of* $A(9)$. We have

$$\int |S_9|^{18+16\lambda}\, d\alpha \ll P^{9+15\lambda+\epsilon} \qquad (\lambda = 1, 2, 3, 4, 5, 6), \tag{56}$$

$$\int |S_9|^{114+32\lambda}\, d\alpha \ll P^{99+31\lambda+\epsilon} \qquad (\lambda = 1, 2, 3, 4, 5, 6), \tag{57}$$

$$\int |S_9|^{306+64\lambda}\, d\alpha \ll P^{285+63\lambda+\epsilon} \qquad (\lambda = 1, 2, \ldots, 7), \tag{58}$$

$$\int |S_9|^{882}\, d\alpha \ll P^{853+\epsilon}, \tag{59}$$

$$\int |S_9|^{882+256\lambda}\, d\alpha \ll P^{853+255\lambda+\epsilon} \qquad (\lambda = 1, 2, \ldots, 8). \tag{60}$$

2.20. *Proof of* $B(9)$. We have

$$\int |C_9|^{20+16\lambda}\, d\alpha \ll P^{10+15\lambda+\epsilon} \qquad (\lambda = 1, \ldots, 5), \tag{61}$$

$$\int |C_9|^{100+32\lambda}\, d\alpha \ll P^{85+31\lambda+\epsilon} \qquad (\lambda = 1, \ldots, 6), \tag{62}$$

$$\int |C_9|^{292+64\lambda}\, d\alpha \ll P^{271+63\lambda+\epsilon} \qquad (\lambda = 1, \ldots, 7), \tag{63}$$

$$\int |C_9|^{740+128\lambda}\, d\alpha \ll P^{712+127\lambda+\epsilon} \qquad (\lambda = 1, \ldots, 8), \tag{64}$$

$$\int |C_9|^{1764+256\lambda}\, d\alpha \ll P^{1738+255\lambda+\epsilon} \qquad (\lambda = 1, \ldots, 9). \tag{65}$$

2.21. *Proof of* $A(10)$. We have

$$\int |S_{10}|^{20+16\lambda}\, d\alpha \ll P^{10+15\lambda+\epsilon} \qquad (\lambda = 1, \ldots, 5), \tag{66}$$

$$\int |S_{10}|^{100+32\lambda}\, d\alpha \ll P^{85+31\lambda+\epsilon} \qquad (\lambda = 1, \ldots, 6), \tag{67}$$

$$\int |S_{10}|^{292+64\lambda}\, d\alpha \ll P^{271+63\lambda+\epsilon} \qquad (\lambda = 1, \ldots, 7), \tag{68}$$

$$\int |S_{10}|^{740+128\lambda}\, d\alpha \ll P^{712+127\lambda+\epsilon} \qquad (\lambda = 1, \ldots, 8), \tag{69}$$

$$\int |S_{10}|^{2020}\, d\alpha \ll P^{1983+\epsilon}, \tag{70}$$

$$\int |S_{10}|^{2020+512\lambda}\, d\alpha \ll P^{1983+511\lambda+\epsilon} \qquad (\lambda = 1, \ldots, 9). \tag{71}$$

2.22. *Proof of* $B(10)$. As an evident consequence of (8) we have

$$\int |C_{10}|^{38}\, d\alpha \ll P^{27+\epsilon}.$$

Then

$$\int |C_{10}|^{38+16\lambda}\, d\alpha \ll P^{27+15\lambda+\epsilon} \quad (\lambda = 1,\dots, 4), \tag{72}$$

$$\int |C_{10}|^{102+32\lambda}\, d\alpha \ll P^{87+31\lambda+\epsilon} \quad (\lambda = 1,\dots, 6), \tag{73}$$

$$\int |C_{10}|^{294+64\lambda}\, d\alpha \ll P^{273+63\lambda+\epsilon} \quad (\lambda = 1,\dots, 7), \tag{74}$$

$$\int |C_{10}|^{742+128\lambda}\, d\alpha \ll P^{714+127\lambda+\epsilon} \quad (\lambda = 1,\dots, 8), \tag{75}$$

$$\int |C_{10}|^{1766+256\lambda}\, d\alpha \ll P^{1730+255\lambda+\epsilon} \quad (\lambda = 1,\dots, 9), \tag{76}$$

$$\int |C_{10}|^{4070+512\lambda}\, d\alpha \ll P^{4025+511\lambda+\epsilon} \quad (\lambda = 1,\dots, 10). \tag{77}$$

3. Applications

In this section I would like to mention some of the possible applications of the theorem.

1. *Tarry's problem.* I hope to discuss this in a later paper.

2. *Waring's problem.* Let $G(k)$ be the least integer s such that

$$N = x_1^k + \dots + x_s^k$$

is solvable in non-negative integers $x_1,\dots, x_s$ for all sufficiently large positive integers N. It was proved by Vinogradow that

$$G(k) < 4k \log k + 8k \log\log k + 12k.$$

If we introduce the result of this paper, and follow the same method as Vinogradow, we can obtain a result which is better than his for $k \leqslant 15{,}000$.

3. *Estimation of trigonometrical sums.* An improvement upon some of the results† due to Vinogradow has been obtained; the details will be given elsewhere later.

4. *Simultaneous Waring's problem.* Let T be the number of solutions of

$$x_1^h + \dots + x_s^h = N_h \quad (h = 1, 2,\dots, k),$$

in positive integers. The result of this paper implies that

$$T = O(P^{s-\frac{1}{2}k(k+1)+\epsilon}),$$

provided that s has the value given in the table in the Introduction. The exact relation between T and a multiple Fourier integral, and some theorems concerning congruences, will be given elsewhere later.

† The results in the papers cited in footnotes † and ‡ on p. 161, and *Bull. de l'Acad. des Sciences de l'U.R.S.S.*, 1938, 399–416.

THE FUNCTIONAL EQUATION FOR EPSTEIN'S ZETA-FUNCTION

By P. R. TAYLOR (*Oxford*)

[Received 28 February 1940]

1. THE functional equation for a general zeta-function of Epstein's type has been found by several methods. Epstein* himself used the theory of transformation of theta-functions, Mordell† has given a proof depending on Poisson's summation formula, and a proof given by Potter‡ depends on the theory of Bessel functions. An expression for an Epstein zeta-function of the form

$$f(s, A, B, C) = \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(Am^2+Bmn+Cn^2)^s}$$

as a rapidly convergent double series which plainly exhibits the functional equation has been given by Kober.§ In this paper I give a proof of the functional equation for $f(s, A, B, C)$ in the special case when the quadratic form is positive-definite. The proof makes use of the functional equation for Riemann's zeta-function, and certain simple properties of hypergeometric functions. I also prove Kober's result by the same method. The proof can presumably be extended to cover those values of A, B, C which make the real part of the quadratic form positive-definite. By repeated application of the method the functional equation for any Epstein zeta-function can be found.

2. We have

$$\int_0^1 \frac{x^{\frac{1}{2}a-1}(1-x)^{\frac{1}{2}b-1}}{\{m^2x+n^2(1-x)\}^{\frac{1}{2}(a+b)}}\,dx = \frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)}{m^a n^b \Gamma\{\frac{1}{2}(a+b)\}} \quad (\Re a,\ \Re b > 0). \tag{1}$$

The integral is evaluated by putting $x' = \dfrac{m^2x}{m^2x+n^2(1-x)}$.

* P. Epstein, 'Zur Theorie allgemeiner Zetafunktionen': *Math. Annalen* 56 (1903), 615–44, and *Math. Annalen*, 63 (1907), 205–16.

† L. J. Mordell, 'The zeta-functions arising from quadratic forms, and their functional equations': *Quart. J. of Math.* (Oxford), 1 (1930), 77–101.

‡ H. S. A. Potter, 'Approximate equations for the Epstein zeta-function': *Proc. London Math. Soc.* (2) 36 (1932), 501–15.

§ H. Kober, 'Transformationsformeln gewisser Besselscher Reihen, Beziehungen zu Zeta-Funktionen', *Math. Zeits.* 39 (1934), 609–24.

Consider

$$\tfrac{1}{2}\int\limits_0^1 x^{\frac{1}{2}a-1}(1-x)^{\frac{1}{2}b-1}\Bigg[\frac{1}{\{m^2x+mnyx^{\frac{1}{2}}(1-x)^{\frac{1}{2}}+n^2(1-x)\}^{\frac{1}{2}(a+b)}}+ \\ +\frac{1}{\{m^2x-mnyx^{\frac{1}{2}}(1-x)^{\frac{1}{2}}+n^2(1-x)\}^{\frac{1}{2}(a+b)}}\Bigg]\,dx. \quad (2)$$

If $|y| < 2$, we may expand the expression in brackets in powers of

$$\frac{mnyx^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}{m^2x+n^2(1-x)},$$

by the binomial theorem, and integrate term by term. This is justified by uniform convergence. Each coefficient of the series can be evaluated by (1), and the value of the integral is found to be

$$\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)}{m^a n^b\Gamma\{\frac{1}{2}(a+b)\}}F(\tfrac{1}{2}a, \tfrac{1}{2}b; \tfrac{1}{2}; \tfrac{1}{4}y^2). \quad (3)$$

Making the substitution $x'^2 = x/(1-x)$ in (2), and putting $b = s$, $a+b = z$, the formula becomes

$$\int\limits_0^\infty x^{s-1}\Bigg[\frac{1}{(m^2+mnyx+n^2x^2)^{\frac{1}{2}z}}+\frac{1}{(m^2-mnyx+n^2x^2)^{\frac{1}{2}z}}\Bigg]\,dx \\ = \frac{\Gamma\{\frac{1}{2}(z-s)\}\Gamma(\frac{1}{2}s)}{m^{z-s}n^s\Gamma(\frac{1}{2}z)}F\{\tfrac{1}{2}(z-s), \tfrac{1}{2}s; \tfrac{1}{2}; \tfrac{1}{4}y^2\}. \quad (4)$$

The integral is absolutely convergent for all values of s, z, y such that $\Re s > 0$, $\Re(z-s) > 0$ and $y^2/4$ is not real and greater than 1. Also the integrand is of bounded variation and continuous in any interval of values of x. Hence Mellin's inversion formula holds.

If we regard z as fixed, this is

$$\frac{1}{(m^2+mnyx+n^2x^2)^{\frac{1}{2}z}}+\frac{1}{(m^2-mnyx+n^2x^2)^{\frac{1}{2}z}} \\ = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{1}{m^{z-s}n^s}\frac{\Gamma\{\frac{1}{2}(z-s)\}\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}z)}F\{\tfrac{1}{2}(z-s), \tfrac{1}{2}s; \tfrac{1}{2}; \tfrac{1}{4}y^2\}x^{-s}\,ds \\ (c,\ \Re z-c > 0). \quad (5)$$

When c, $\Re z-c > 1$, we can sum for m and n from 1 to ∞, and invert the order of integration and summation, provided that all the integrals with which we deal are absolutely convergent. The following

lemma shows that this is so, and it also justifies every translation of the line of integration which is used later.

LEMMA. *If y is real, $0 < y < 2$, and $\Re b_0 > 0$, as $|t| \to \infty$, then*

$$F\{\tfrac{1}{2}(a_0+it), \tfrac{1}{2}(b_0-it); \tfrac{1}{2}; \tfrac{1}{4}y^2\} = O\left\{|t|^A \exp\left[|t| \tan^{-1}\frac{y}{(4-y^2)^{\frac{1}{2}}}\right]\right\}. \quad (6)$$

This is a particular case of a result due to Watson.* Since Watson's analysis is very complicated, and leads to results which are more than sufficient for our purpose, it seems convenient to insert a simple proof of a blunter result.

In (2), put $y = i\lambda$, where λ is real, and put $m = n = 1$. If $1+i\lambda x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} = re^{i\theta}$, then

$$|\{1+i\lambda x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\}^{\sigma+it}| = r^{\sigma}e^{-t\theta}.$$

Now it is easily seen that, with a fixed λ, $|\theta|$ attains its maximum value in the interval $(0,1)$ when $x = \frac{1}{2}$ and that this value is $|\tan^{-1}\frac{1}{2}\lambda|$. Hence, putting $a = a_0+it$, $b = b_0+it$, making $|t| \to \infty$, and using the asymptotic formula for the Γ-function, we find

$$F(\tfrac{1}{2}a, \tfrac{1}{2}b; \tfrac{1}{2}; -\tfrac{1}{4}\lambda^2) = O(|t|^A \exp[|t\tan^{-1}\tfrac{1}{2}\lambda|]) \quad (\Re a_0, \Re b_0 > 0).$$

Using the formula

$$F(a,b;c;x) = (1-x)^{-b}F\left(c-a,b;c;\frac{x}{x-1}\right),$$

we get

$$F\left\{\tfrac{1}{2}(a_0'-it); \tfrac{1}{2}(b_0+it); \tfrac{1}{2}; \frac{\lambda^2}{4+\lambda^2}\right\} = O(|t|^A \exp[|t\tan^{-1}\tfrac{1}{2}\lambda|]),$$

if $\Re a_0'$ is sufficiently small, and (6) follows, with this restriction on $\Re a_0$, by putting $\lambda^2/(4+\lambda^2) = \frac{1}{4}y^2$. The general result follows by observing that

$$a\{F(a+1,b;c;x)-F(a,b;c;x)\} = b\{F(a,b+1;c;x)-F(a,b;c;x)\}.$$

This shows that, if (6) holds for a_0, it also holds for a_0+2.

Let

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) = \xi(s),$$

and

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\left[\sum_{m,n}{}'' \frac{1}{(m^2+mnyx+n^2x^2)^{\frac{1}{2}s}}\right] = \xi_{x,y}(s),$$

where $\sum''$ denotes a summation for values of m and n from $-\infty$ to ∞,

* G. N. Watson, *Cambridge Phil. Trans.* 22 (1918), 277–308.

missing out zero values. Then, summing (5) for values of m and n from 1 to ∞, we get

$$\xi_{x,y}(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \xi(s)\xi(z-s)F\{\tfrac{1}{2}s, \tfrac{1}{2}(z-s); \tfrac{1}{2}; \tfrac{1}{4}y^2\}x^{-s}\,ds \quad (c, \Re z-c > 1). \tag{7}$$

Move the contour to $\Re s = c'$, where $0 < c' < 1$. We pass the pole of $\xi(s)$ at $s = 1$ with residue

$$x^{-1}\xi(z-1)F\{\tfrac{1}{2}, \tfrac{1}{2}(z-1); \tfrac{1}{2}; \tfrac{1}{4}y^2\} = x^{-1}(1-\tfrac{1}{4}y^2)^{-\frac{1}{2}(z-1)}\xi(z-1).$$

This process enables us to continue (7) to all values of z such that $\Re z-c' > 1$. Provided that $\Re z$ is now made small enough, the line of integration may be moved to the right to pass over the pole at $s = z-1$ with residue

$$-x^{-(z-1)}(1-\tfrac{1}{4}y^2)^{-\frac{1}{2}(z-1)}\xi(z-1).$$

Hence, if $0 < c' < 1$, $0 < \Re z-c' < 1$,

$$\xi_{x,y}(z)-(x^{-1}+x^{-(z-1)})(1-\tfrac{1}{4}y^2)^{-\frac{1}{2}(z-1)}\xi(z-1)$$
$$= \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \xi(s)\xi(z-s)F\{\tfrac{1}{2}s, \tfrac{1}{2}(z-s); \tfrac{1}{2}; \tfrac{1}{4}y^2\}x^{-s}\,ds. \tag{8}$$

Using the functional equation

$$\xi(s) = \xi(1-s)$$

for both ξ-functions, and the formula

$$F(a,b;c;x) = (1-x)^{c-a-b}F(c-a, c-b; c; x),$$

we find that the integrand in (8) is unaltered in form when z is replaced by $2-z$ and c' by $1-c'$. We infer that

$$\xi_{x,y}(z)-\{x^{-1}+x^{-(z-1)}\}(1-\tfrac{1}{4}y^2)^{-\frac{1}{2}(z-1)}\xi(z-1)$$
$$= x^{-1}(1-\tfrac{1}{4}y^2)^{\frac{1}{2}-\frac{1}{2}z}[\xi_{x^{-1},y}(2-z)-(x+x^{1-z})(1-\tfrac{1}{4}y^2)^{-\frac{1}{2}+\frac{1}{2}z}\xi(1-z)].$$

Transposing the last two terms on each side and noting that

$$\xi_{x^{-1},y}(z) = x^z\xi_{x,y}(z),$$

we find that

$$\pi^{-s}\Gamma(s)f(s,1,xy,x^2) = \{x^2(1-\tfrac{1}{4}y^2)\}^{\frac{1}{2}-s}\pi^{1-s}\Gamma(1-s)f(1-s,1,xy,x^2),$$

the functional equation.

Watson's result enables us to prove the functional equation for $f(s,A,B,C)$ by this method when A, B, C have certain complex values.

3. In (7) move the contour to $\mathfrak{R}s = c'$, where $c' < 0$. We get

$$\xi_{x,y}(z)-\xi(z-1)x^{-1}(1-\tfrac{1}{4}y^2)^{\frac{1}{2}-\frac{1}{2}z}+\xi(z)$$

$$=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\xi(s)\xi(z-s)F\{\tfrac{1}{2}s,\tfrac{1}{2}(z-s);\tfrac{1}{2};\tfrac{1}{4}y^2\}x^{-s}\,ds \quad (c'<0,\ \mathfrak{R}z-c'>1). \tag{9}$$

If $f(z)$ denotes either side of this equation, we see that for any value of z, c' can always be chosen to satisfy the conditions imposed on it, so that $f(z)$ is an integral function of z.

Using the functional equation for the first ξ-function, and the formula

$$F(a,b;c;x) = (1-x)^{-b}F\left(c-a,b;c;\frac{x}{x-1}\right),$$

(9) becomes

$$f(z)=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\xi(1-s)\xi(z-s)\times$$

$$\times F\left\{\tfrac{1}{2}(1-s),\tfrac{1}{2}(z-s);\tfrac{1}{2};\frac{y^2}{y^2-4}\right\}(1-\tfrac{1}{4}y^2)^{-\frac{1}{2}(z-s)}x^{-s}\,ds,$$

i.e., putting $c''=1-c'$, so that $c''>1$, $\mathfrak{R}z+c''>2$,

$$f(z)=\frac{1}{2\pi i}\int_{c''-i\infty}^{c''+i\infty}\xi(s)\xi(z-1+s)\times$$

$$\times F\left\{\tfrac{1}{2}s,\tfrac{1}{2}(z-1+s);\tfrac{1}{2};\frac{y^2}{y^2-4}\right\}(1-\tfrac{1}{4}y^2)^{-\frac{1}{2}(z-1+s)}x^{s-1}\,ds.$$

Writing $\xi(s)=\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s)$, and expanding both the ζ-functions as Dirichlet series, we get

$$f(z)=\sum_{m,n=1}^{\infty}\frac{1}{2\pi i}\int_{c''-i\infty}^{c''+i\infty}(\pi mn)^{-s}\pi^{-\frac{1}{2}(z-1)}n^{-(z-1)}\Gamma(\tfrac{1}{2}s)\Gamma\{\tfrac{1}{2}(z-1+s)\}\times$$

$$\times F\left\{\tfrac{1}{2}s,\tfrac{1}{2}(z-1+s);\tfrac{1}{2};\frac{y^2}{y^2-4}\right\}(1-\tfrac{1}{4}y^2)^{-\frac{1}{2}(z-1+s)}x^{s-1}\,ds. \tag{10}$$

The evaluation of this expression obviously turns on the evaluation of the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\nu)F(\tfrac{1}{2}s,\tfrac{1}{2}s+\nu;\tfrac{1}{2};-\tfrac{1}{4}Y^2)X^{-s}\,ds \quad (c,\ c+2\mathfrak{R}\nu>1). \tag{11}$$

If $|Y| < 2$, this integral is

$$\sum_{k=0}^{\infty} \frac{(-Y^2)^k}{(2k)!} \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \Gamma(\tfrac{1}{2}s+k)\Gamma(\tfrac{1}{2}s+\nu+k)X^{-s}\, ds$$

$$= \sum_{k=0}^{\infty} \frac{(-X^2Y^2)^k}{(2k)!} \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\nu)X^{-s}\, ds$$

$$= 4X^{\nu}K_{\nu}(2X)\cos(XY),$$

in the usual Bessel-function notation.*

Putting in this value for the integrals in (10), we get Kober's result

$$f(z) = 4x^{-1}\{x(1-\tfrac{1}{4}y^2)^{\frac{1}{2}}\}^{\frac{1}{2}(1-z)}\times$$

$$\times \sum_{m,n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{2}(1-z)} \cos\left(\frac{\pi mny}{x}\right) K_{\frac{1}{2}(1-z)}\left(\frac{2\pi mn(1-\frac{1}{4}y^2)^{\frac{1}{2}}}{x}\right).$$

* See, e.g., Titchmarsh, *Theory of Fourier Integrals* (Oxford, 1937), § 7.9 (12).

RATIO TESTS FOR DOUBLE POWER SERIES

By P. J. DANIELL (*Sheffield*)

[Received 28 March 1940]

GIVEN a double power series of non-negative terms

$$\sum c_{s,t}\, x^s y^t \quad (x,\, y,\, c_{s,t} > 0)$$

the problem is to determine the region R of correlated variables (x,y) such that the series is convergent if (x,y) belongs to R and divergent when it belongs neither to R nor to its boundary. A formula has been given by Lemaire* which is that for a fixed value of $k = y/x$ the series is convergent for $x < r_1$, divergent for $x > r_1$ if

$$\frac{1}{r_1} = \overline{\lim}\, \sqrt[n]{(c_{s,t}\, k^t)} \quad (n = s+t).$$

In the first place we shall obtain a general test (Theorem 3) of the same type though expressed differently. This defines R as the set of (x,y) $(x,\, y > 0)$ such that for all p $(0 \leqslant p \leqslant 1;\ q = 1-p)$

$$p \log x + q \log y + f(p) < 0.$$

The function $f(p)$ is called the *envelope function* and R is the region bounded in a certain sense by the 'envelope' of the family of curves

$$p \log x + q \log y + f(p) = 0.$$

The series will be divergent when (x,y) is such that for at least one p

$$p \log x + q \log y + f(p) > 0.$$

This paper does not consider convergence on the boundary of R.

It is clear that, if (x,y) belongs to R, so also must (x',y') if $0 < x' \leqslant x$, $0 < y' \leqslant y$. We shall also show that R is an open set.

Various tests are developed and finally (Test E) it is shown that under certain conditions

$$f(p) = \lim p \log(c_{s+1,t}/c_{s,t}) + \lim q \log(c_{s,t+1}/c_{s,t}),$$

where $n = s+t$ and s/n tends to p as n increases indefinitely. Throughout the paper it is to be understood that $n = s+t$, that $p+q = 1$, and that $0 \leqslant p \leqslant 1$.

Although the theory applies to power series it can of course be made applicable to any double series of non-negative terms by putting $x = y = 1$.

* Lemaire, *Bull. des Sci. Math.* 20 (1896), 286; Bromwich, *Infinite Series* (1908), 504.

Though attention is confined to double series, all tests can be extended immediately to series of any bounded multiplicity.

LEMMA. *The double series of non-negative terms* $\sum a_{s,t}$ *is*

(i) *convergent if* $\overline{\lim}_n \sqrt[n]{(a_{s,t})} < 1$; (ii) *divergent if* $\overline{\lim}_n \sqrt[n]{(a_{s,t})} > 1$.

The upper limit used here is not Pringsheim's double upper limit with regard to both s and t but is the limit, as n_1 increases indefinitely, of the upper bound of terms of the double sequence for all s, t such that $s+t \geqslant n_1$. In other words,

$$\overline{\lim_n}\, \phi(s,t) = \overline{\lim_n}\, M_n,$$

where M_n is the maximum value of $\phi(s,t)$ for each given $n = s+t$.

Proof. In case (i) let the upper limit be ρ (<1) and let $r = \frac{1}{2}(1+\rho)$. Then $r < 1$ and $a_{s,t} < r^n$ when $n = s+t \geqslant n_1$. Therefore,

$$\sum_{s+t\geqslant n_1} a_{s,t} < \sum_{n=n_1}^{\infty} (n+1)r^n$$

is convergent.

In case (ii) it is possible to find an unending sequence of terms for which $a_{s,t} > 1$ and the series must be divergent.

For our purposes it is necessary to analyse this lemma in more detail.

DEFINITION 1. When $0 \leqslant p \leqslant 1$, define

$$\alpha(p) = \lim_{\substack{n_1=\infty\\ \epsilon=0}} \max_{\substack{n\geqslant n_1\\ |s/n-p|\leqslant\epsilon}} \left(\frac{1}{n}\log a_{s,t}\right),$$

$$A(p',p'') = \overline{\lim_n}\left(\frac{1}{n}\log a_{s,t}\right) \quad (np' \leqslant s \leqslant np'').$$

The term 'max' is used here to denote the upper bound.

We note that $\alpha(p)$ is the limit of a monotone double sequence and therefore it is also the repeated limit as ϵ tends to 0 of the limit as n_1 tends to infinity. In other words, if p belongs to each of a sequence of intervals (p',p''), where $p''-p'$ tends to 0, then $\alpha(p)$ is the limit of $A(p',p'')$. If $p \neq 0$, 1 we take $p' < p < p''$. If $p = 0$ we must take $p' = 0$ while, if $p = 1$, we take $p'' = 1$.

THEOREM 1. *In the closed interval* $(0,1)$, $\alpha(p)$ *attains its upper bound which is*

$$A(0,1) = \overline{\lim_n}\left(\frac{1}{n}\log a_{s,t}\right).$$

Proof. If $a < b < c$,

$$A(a,c) = \max[A(a,b), A(b,c)].$$

Subdivide the interval $(0, 1)$ into $2, 4,\ldots, 2^r,\ldots$ sub-intervals and choose them successively inside one another so that for each in turn

$$A(p', p'') = A(0,1).$$

The intervals define a point p_1 contained in all of them. For all these intervals $A(p', p'') = A(0,1)$ and therefore in the limit

$$\alpha(p_1) = A(0,1).$$

Also, if p is any point and if (p', p'') contains p,

$$A(p', p'') \leqslant A(0,1) = \alpha(p_1).$$

Therefore, in the limit, $\quad \alpha(p) \leqslant \alpha(p_1).$

The theorem is proved and we can now express our lemma in a different form.

THEOREM 2. *The double series of non-negative terms* $\sum a_{s,t}$ *is*

(i) *convergent if* $\alpha(p) < 0$ *for all* p $(0 \leqslant p \leqslant 1)$;

(ii) *divergent if* $\alpha(p) > 0$ *for some* p,

where $\alpha(p)$ *is given by definition* 1, *namely*

$$\alpha(p) = \lim_{\substack{n_1=\infty \\ \epsilon=0}} \max_{\substack{n>n_1 \\ |s/n-p|\leqslant\epsilon}} \left(\frac{1}{n}\log a_{s,t}\right).$$

We now apply this to power series.

THEOREM 3. *The double power series*

$$\sum c_{s,t}x^s y^t \quad (c_{s,t} \geqslant 0,\, x > 0,\, y > 0)$$

is

(i) *convergent at* (x,y) *if for all* p $(0 \leqslant p \leqslant 1)$

$$p\log x + q\log y + f(p) < 0;$$

(ii) *divergent at* (x,y) *if for some* p $(0 \leqslant p \leqslant 1)$

$$p\log x + q\log y + f(p) > 0,$$

where

$$f(p) = \lim_{\substack{n_1=\infty \\ \epsilon=0}} \max_{\substack{n\geqslant n_1 \\ |s/n-p|\leqslant\epsilon}} \left(\frac{1}{n}\log c_{s,t}\right).$$

Proof. Put

$$a_{s,t} = c_{s,t}x^s y^t,$$

$$\frac{1}{n}\log a_{s,t} = \frac{s}{n}\log x + \frac{t}{n}\log y + \frac{1}{n}\log c_{s,t}.$$

If $C(p',p'')$ is defined for $c_{s,t}$ as $A(p',p'')$ is defined for $a_{s,t}$, then

$$|A(p',p'')-p'\log x-q'\log y-C(p',p'')| \leqslant (p''-p')[|\log x|+|\log y|].$$

Proceeding to upper bounds and then to limits we obtain

$$\alpha(p) = p\log x+q\log y+f(p).$$

Hence the theorem is a corollary of Theorem 2.

DEFINITION 2. Let $f(p)$ be defined as in Theorem 3. We call this function the *envelope function*. The set of points (x,y) for which condition (i) of Theorem 3 is satisfied and for which $x > 0$, $y > 0$ is called the *region R corresponding to $f(p)$* or the *open domain of convergence.*

If (x_1,y_1) belongs to R, since $\alpha(p)$ attains its upper bound, there must exist a positive δ such that for all p,

$$p\log x_1+q\log y_1+f(p) \leqslant -2\delta.$$

Choose a circle with centre (x_1,y_1) of such a radius that for every point within it

$$|\log x/x_1| < \delta, \qquad |\log y/y_1| < \delta.$$

Then for all p

$$p\log x+q\log y+f(p) \leqslant -2\delta+(p+q)\delta \leqslant -\delta < 0.$$

Hence (x,y) also belongs to R. In other words, R is an open set.

Degenerate case. If there is some value of p for which $f(p) = \infty$, then R reduces to the null set and the series is divergent if $x > 0$, $y > 0$.

Special case A. This is a case which is by no means degenerate but in which the region R is particularly simple.

If, for all s,t,

$$(c_{s,t})^n \leqslant (c_{n,0})^s(c_{0,n})^t \quad (n = s+t),$$

then the region R of convergence consists of the points (x,y) such that $0 < x < a$, $0 < y < b$ where a, b are the radii of convergence of the single series

$$\sum c_{n,0}x^n, \qquad \sum c_{0,n}y^n,$$

respectively. In particular, if $c_{n,0} = c_{0,n} = 1$, $c_{s,t} \leqslant 1$, then R is the open square $0 < x < 1$, $0 < y < 1$.

Proof. Let $\quad \sqrt[n]{c_{n,0}} = \alpha_n, \qquad \sqrt[n]{c_{0,n}} = \beta_n.$

Then $\quad \overline{\lim}\,\alpha_n = 1/a, \qquad \overline{\lim}\,\beta_n = 1/b.$

Case (i). If $0 < x < a$, $0 < y < b$, choose

$$x_1 = \tfrac{1}{2}(x+a), \qquad y_1 = \tfrac{1}{2}(y+b)$$

so that $x < x_1 < a$, $y < y_1 < b$. Then

$$c_{s,t}x^s y^t \leqslant (\alpha_n x)^s(\beta_n y)^t.$$

But we can choose n_1 so that, if $n \geqslant n_1$, $\alpha_n < 1/x_1$, $\beta_n < 1/y_1$. Hence

$$c_{s,t}x^s y^t \leqslant (x/x_1)^s(y/y_1)^t,$$

and the power series converges like a double geometric series.

Case (ii). If $x > a$ or $y > b$ (or both), let us suppose that the former is true. Then

$$\sum c_{s,t}x^s y^t \geqslant \sum c_{n,0}x^n$$

which is divergent.

In terms of Theorem 3 this is the case where

$$f(0) = -\log b, \qquad f(1) = -\log a, \qquad f(p) \leqslant -(p\log a + q\log b).$$

The region R is that bounded by the lines

$$p\log x/a + q\log y/b = 0 \quad (0 \leqslant p \leqslant 1).$$

Test B. If for all rational p $(0 \leqslant p \leqslant 1)$,

$$\lim_{\substack{n=\infty \\ s=pn}} \frac{1}{n}\log c_{s,t} = f_1(p)$$

exists, and if the convergence is uniform with respect to p, then the envelope function $f(p)$ is the upper limiting function of $f_1(p)$, that is,

$$f(p) = \bar{f}_1(p) = \lim_{\epsilon=0} \max_{|p-p'|\leqslant\epsilon} f_1(p') \quad (p' \text{ rational}).$$

Proof. By hypothesis, given any $\delta > 0$ we can find n_0 so that for all rational p', if $s = p'n$, $n \geqslant n_0$,

$$\left|\frac{1}{n}\log c_{s,t} - f_1(p')\right| \leqslant \delta.$$

Now, if for two sets of quantities $a(\alpha)$, $b(\alpha)$, $|a(\alpha)-b(\alpha)| \leqslant \delta$, then by considering two inequalities,

$$|\max a(\alpha) - \max b(\alpha)| \leqslant \delta.$$

Therefore, in our case, if $n_1 \geqslant n_0$ for all p, ϵ,

$$\left|\max_{\substack{n\geqslant n_1 \\ |p-s/n|\leqslant\epsilon}} \left(\frac{1}{n}\log c_{s,t}\right) - \max_{|p-p'|\leqslant\epsilon} f_1(p')\right| \leqslant \delta.$$

Taking the double monotone limits as n_1 increases indefinitely, ϵ tends to 0, we see that

$$|f(p)-\bar{f}_1(p)| \leqslant \delta.$$

This is true for all positive δ, so that

$$f(p) = \bar{f}_1(p)$$

and the theorem is proved.

This test is more of a stepping-stone than useful in itself. On the one hand it uses only rational values of p, and on the other hand values for different p are completely independent.

Test C. If $\gamma(\xi, \eta)$ is so defined for all non-negative ξ, η whose sum is an integer n that $c_{s,t} = \gamma(s,t)$ and if

$$\lim_{n=\infty} \frac{1}{n} \log \gamma(pn, qn) = f_1(p),$$

the convergence being uniform with respect to p, then the envelope function is $f(p)$, the upper limiting function of $f_1(p)$, that is,

$$f(p) = \bar{f}_1(p).$$

In particular, if for each value of $n = \xi+\eta$, $\gamma(\xi, \eta)$ is a continuous function of ξ, then $f_1(p)$ is continuous and $f(p) = f_1(p)$.

The proof is exactly as for test B and we have avoided the restriction to rational p. The particular result follows from the well-known property of the limit of a uniformly convergent sequence of continuous functions.

Test D. (*Total ratio test.*) If $\gamma(\xi, \eta)$ is so defined for all non-negative ξ, η whose sum is an integer n that $c_{s,t} = \gamma(s,t)$ and that it is continuous for each fixed n, and if

$$\lim_{n=\infty} [\log \gamma(np+p, nq+q) - \log \gamma(np, nq)] = f(p),$$

the convergence being uniform with respect to p, then the envelope function is $f(p)$.

Proof. By hypothesis, given $\delta > 0$ we can choose n_1 so that, if $n \geqslant n_1$,

$$|\log \gamma(np+p, nq+q) - \log \gamma(np, nq) - f(p)| \leqslant \tfrac{1}{2}\delta.$$

Therefore, by successive additions,

$$\begin{aligned} |\log \gamma(np, nq) - \log \gamma(n_1 p, n_1 q) - (n-n_1) f(p)| &\leqslant \tfrac{1}{2}(n-n_1)\delta \\ &< \tfrac{1}{2} n\delta. \end{aligned}$$

Since $\gamma(n_1 p, n_1 q)$ is continuous in p, its modulus has an upper bound, say a_1. Similarly, $|f(p)|$ has an upper bound, say a. Then

$$\left|\frac{1}{n}\log\gamma(np,nq)-f(p)\right| \leqslant \tfrac{1}{2}\delta+(a_1+n_1 a_2)/n.$$

Put $A = a_1+n_1 a_2$ and choose

$$n_2 = \max(n_1, 2A/\delta).$$

Then, if $n \geqslant n_2$, $$\left|\frac{1}{n}\log\gamma(np,nq)-f(p)\right| \leqslant \delta.$$

This proves the theorem by test C and its particular case.

Definition 3. We say that

$$\lim_{n=\infty}\phi(s,t) = f(p)$$

with *interlocked* uniform convergence with respect to p if given any $\delta > 0$ we can find n_0 independent of p such that

$$|\phi(s,t)-f(p)| < \delta,$$

provided that $n = s+t \geqslant n_0$, $|s-pn| \leqslant 1$.

Given such interlocked uniform convergence, define $\psi(\xi,\eta)$ to be equal to $\phi(\xi,\eta)$ at points s, t of the integral lattice, while along each diagonal line $\xi+\eta = n$ (n integral and constant) let ψ be linearly interpolated between lattice points. Then, for each n, ψ is continuous. Also, if $s = [pn]$, the greatest integer less than or equal to pn, $\psi(pn,qn)$ lies between $\psi(s,t)$ and $\psi(s+1,t-1)$, where $t = n-s$. Hence, by the interlocking condition, $\psi(pn,qn)$ tends to $f(p)$ uniformly and $f(p)$ is continuous. This definition allows us to obtain two immediate corollaries of tests B and D.

Test B′. If $$\lim\frac{1}{n}\log c_{s,t} = f(p)$$

with interlocked uniform convergence, then the envelope function is $f(p)$ and this is now continuous.

Test D′. If, when s, t are integral,

$$\phi(s,t) = \log c_{s,t},$$

if along each diagonal $\xi+\eta = \text{constant } (n)$, $\phi(\xi,\eta)$ is linearly interpolated between lattice points, and if

$$\lim_{n=\infty}[\phi(np+p,nq+q)-\phi(np,nq] = f(p)$$

uniformly with respect to p, then $f(p)$ is the envelope function.

Test E. (*Partial ratio test.*) If $c_{s,t} > 0$ for all s, t, and if

$$\lim_{n=\infty} p\log(c_{s+1,t}/c_{s,t}) = g(p), \qquad \lim_{n=\infty} q\log(c_{s,t+1}/c_{s,t}) = h(p)$$

each with interlocked uniform convergence, then

$$f(p) = g(p)+h(p)$$

is the envelope function and the region R of convergence of the double power series $\sum c_{s,t}x^sy^t$ $(x > 0,\ y > 0)$ is the set R of points (x,y) such that for all p $(0 \leqslant p \leqslant 1,\ q = 1-p)$

$$p\log x+q\log y+f(p) < 0.$$

Proof. Define $\phi(\xi,\eta)$ by linear interpolation of $\log c_{s,t}$ as in test D′. Let

$$\Delta(\xi,\eta) = \phi(\xi+p,\eta+q)-\phi(\xi,\eta).$$

Then

$$\begin{aligned}\Delta_1 &= \Delta(s,t) = \phi(s+p,t+q)-\phi(s,t)\\ &= p[\phi(s+1,t)-\phi(s,t)]+q[\phi(s,t+1)-\phi(s,t)],\\ \Delta_2 &= \Delta(s+1,t-1)\\ &= p[\phi(s+2,t-1)-\phi(s+1,t-1)]+q[\phi(s+1,t)-\phi(s+1,t-1)],\\ \Delta_3 &= \Delta(s+q,t-q)\\ &= p[\phi(s+1,t)-\phi(s,t)]+q[\phi(s+1,t)-\phi(s+1,t-1)].\end{aligned}$$

Let A_1, A_2, A_3 be the points (s,t), $(s+1,t-1)$, $(s+q,t-q)$. Then Δ_1, Δ_2, Δ_3 are the increases of $\phi(\xi,\eta)$ from A_1, A_2, A_3 on the diagonal $\xi+\eta = n$ to their oblique parallel projections on the next diagonal $n+1$, the projecting lines having the direction ratios $p:q$.

Let $s = [pn]$, the integral part of pn. Then the points $P = (pn,qn)$, A_3 both lie on the diagonal segment A_1A_2. In each of the segments A_1A_3, A_3A_2, $\Delta(\xi,\eta)$ varies linearly. If P lies on A_1A_3, $\Delta(pn,qn)$ has a value between Δ_1 and Δ_3. If P lies on A_3A_2, $\Delta(pn,qn)$ lies between Δ_3 and Δ_2. Now, by the hypothesis of uniform interlocked convergence, given $\delta > 0$ we can find n_0 so that, if $n \geqslant n_0$,

$$|\Delta_1-f(p)| < \delta, \qquad |\Delta_2-f(p)| < \delta, \qquad |\Delta_3-f(p)| < \delta.$$

Hence

$$|\Delta(pn,qn)-f(p)| = |\phi(pn+p,qn+q)-\phi(pn,qn)-f(p)| < \delta.$$

Thus the conditions of test D′ are satisfied and the theorem is proved.

Illustrations.

(i) $$\sum_{m=0}^{\infty} x^{\alpha m}y^{\beta m} = (1-x^{\alpha}y^{\beta})^{-1} \quad (\alpha, \beta > 0).$$

Unless $\alpha = \beta = 1$, none of the special tests are applicable, but, if we use Theorem 3, $f(p) = -\infty$ except when $p = \alpha/(\alpha+\beta)$ and then $f(p) = 0$. The region R of convergence is the set of points (x, y) for which $\alpha \log x + \beta \log y < 0$.

(ii) $$\sum_{s,t} \binom{s+t}{s}^{\alpha} x^s y^t.$$

If $\alpha \leqslant 0$, since $\binom{s+t}{s} \geqslant 1$, the series falls into special case A and the region R of convergence is the set for which $0 < x < 1$, $0 < y < 1$.

If $\alpha = 0$, the series is

$$\sum x^s y^t = (1-x)^{-1}(1-y)^{-1}.$$

If $\alpha = -1$, the series is

$$S = \sum \frac{s!\,t!}{(s+t)!} x^s y^t.$$

Then, if $0 < x < 1$, $0 < y < 1$,

$$(x+y-xy)S = x(1-x)^{-1}+y(1-y)^{-1}+xyS',$$

$$S' = \sum \frac{s!\,t!}{(s+t+1)!} x^s y^t.$$

This series is also convergent for $0 < x < 1$, $0 < y < 1$, and

$$(x+y-xy)S' = \log(1-x)^{-1}+\log(1-y)^{-1}.$$

Thus

$$S = \frac{x+y-2xy}{(1-x)(1-y)(x+y-xy)} + \frac{xy}{(x+y-xy)^2}\log(1-x)^{-1}(1-y)^{-1}.$$

If $\alpha > 0$, the series does not fall into the special case A but it satisfies the partial ratio test. We have

$$p\log(c_{s+1,t}/c_{s,t}) = -\alpha p \log\frac{s+1}{n+1} \to -\alpha p \log p.$$

If $s = pn+\theta$, then

$$\log p - \log\frac{s+1}{n+1} = -\log\frac{s+1}{pn+p}$$

$$= -\log\left(1+\frac{\theta+q}{pn+p}\right).$$

If $|\theta| \leqslant 1$, this expression lies between $-\log\{1+2/(pn+p)\}$ and

$-\log\{1-1/(n+1)\}$ and therefore between $-2/p(n+1)$ and $2/(n+1)$ $(n > 2)$. Therefore,

$$\left|p\log p - p\log\frac{s+1}{n+1}\right| \leqslant \frac{2}{n+1}.$$

We see that the factor p is necessary to preserve the uniformity of the interlocked convergence down to $p = 0$. Similar reasoning applies to the other ratio, and the envelope function is

$$f(p) = g(p)+h(p) = -\alpha(p\log p+q\log q).$$

The region R of convergence is given by the condition that, for all p,

$$p\log(x/p^{\alpha})+q\log(y/q^{\alpha}) < 0.$$

The envelope of the family of curves

$$p\log(xp^{-\alpha})+q\log(yq^{-\alpha}) = 0 \quad (p+q = 1)$$

is the curve $$x^{1/\alpha}+y^{1/\alpha} = 1,$$

and it is seen that, if $\alpha > 0$ and $x^{1/\alpha}+y^{1/\alpha} < 1$, then the conditions for convergence are satisfied. On the other hand, if $\alpha < 0$, it must be remembered that the curve $x^{1/\alpha}+y^{1/\alpha} = 1$ has asymptotes which are parallel to the axes and pass through $x = 1$, $y = 1$. The region R is always bounded to the right and above by all possible curves of the family touching the envelope. Hence, when $\alpha < 0$, as we have seen already, the region R of convergence is the square $0 < x < 1$, $0 < y < 1$. When $\alpha > 0$, this difficulty does not arise, and the series is convergent when $$x^{1/\alpha}+y^{1/\alpha} < 1.$$

When $\alpha = 1$, the series is

$$\sum \frac{(s+t)!}{s!\,t!}x^s y^t \doteq (1-x-y)^{-1} \quad (x+y < 1).$$

If $\alpha = 2$ and if $\sqrt{x}+\sqrt{y} < 1$, the series is

$$S = \sum \left[\frac{(s+t)!}{s!\,t!}\right]^2 x^s y^t$$

$$= \frac{1}{2\pi i}\sum_n \int (\sqrt{x}+z\sqrt{y})^n(\sqrt{x}+z^{-1}\sqrt{y})^n\,dz/z,$$

where γ is the circle of unit radius with centre 0 in the complex plane. Hence,

$$S = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{1-x-y-2\sqrt{(xy)}\cos\theta}$$

$$= [(1-x-y)^2-4xy]^{-\frac{1}{2}}$$

$$= [(1-\sqrt{x}-\sqrt{y})(1-\sqrt{x}+\sqrt{y})(1+\sqrt{x}-\sqrt{y})(1+\sqrt{x}+\sqrt{y})]^{-\frac{1}{2}}.$$

ON FRACTIONAL INTEGRALS AND DERIVATIVES

By H. KOBER (*Birmingham*)

[Received 2 May 1940]

1. Introduction

THE fundamental theorems on fractional integrals and derivatives were given by Weyl and Hardy-Littlewood.* In the first part of this paper (§§ 2, 3) I shall extend some results, given by them on fractional integrals, over a wider range and shall prove some formulae on integration by parts for the infinite interval; the corresponding formula for the finite interval was given by Love and Young.† Of course the fundamental inequality‡ on which the Hardy-Littlewood theory rests cannot be extended over a wider range, as they prove in their paper; but it can be split up into two inequalities which imply each other; and there is no great difficulty in extending each of them over a much wider range, and these formulae only are needed for the theory of fractional integration.

I introduce a complex parameter η and deal with the operators

$$g^+_{\eta,\alpha}(z) = I^+_{\eta,\alpha}f = \{\Gamma(\alpha)\}^{-1}z^{-\eta-\alpha}\int_0^z (z-t)^{\alpha-1}t^{\eta}f(t)\,dt, \tag{1.1}$$

$$h^-_{\eta,\alpha}(z) = K^-_{\eta,\alpha}f = \{\Gamma(\alpha)\}^{-1}z^{\eta}\int_z^\infty (t-z)^{\alpha-1}t^{-\eta-\alpha}f(t)\,dt, \tag{1.2}$$

$$g^-_{\eta,\alpha}(z) = I^-_{\eta,\alpha}f = \{\Gamma(\alpha)\}^{-1}z^{-\eta-\alpha}\int_z^\infty (t-z)^{\alpha-1}t^{\eta}f(t)\,dt, \tag{1.3}$$

$$h^+_{\eta,\alpha}(z) = K^+_{\eta,\alpha}f = \{\Gamma(\alpha)\}^{-1}z^{\eta}\int_0^z (z-t)^{\alpha-1}t^{-\eta-\alpha}f(t)\,dt. \tag{1.4}$$

* H. Weyl, *Vierteljahrsschr. d. Naturf. Ges.*, Zürich, 62 (1917), 296–302. G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2) 24 (1925), xxxvii–xli (referred to as HL. 1), *Math. Zeits.* 27 (1928), 565–606 (HL. 2); G. H. Hardy, *Messenger of Math.* 47 (1918), 145–50.

† *Proc. London Math. Soc.* (2) 44 (1938), 1–28; E. R. Love, *Proc. London Math. Soc.* (2) 44 (1938), 363–97. The formula for the infinite interval was suggested to me by Dr. A. Erdélyi.

‡ G. H. Hardy, J. E. Littlewood, G. Pólya, *Proc. London Math. Soc.* (2) 25 (1926), 265–82 (HLP. 1), and Theorems 3 and 6 of HL. 2 or Theorem 382 of *Inequalities* by Hardy-Littlewood-Pólya (Cambridge, 1934).

The functions $z^{\eta+\alpha}g^{+}_{\eta,\alpha}(z)$ and $z^{-\eta}h^{+}_{\eta,\alpha}(z)$ are the 'Riemann-Liouville integrals of order α' of $t^{\eta}f(t)$ or $t^{-\eta-\alpha}f(t)$ respectively with 'origin 0', while the functions $z^{\eta+\alpha}g^{-}_{\eta,\alpha}(z)$ and $z^{-\eta}h^{-}_{\eta,\alpha}(z)$ are the 'Weyl integrals of order α'.

By means of the parameter η we get both a simple form and a generalization of some well-known results. Among other properties we have

$$\int\limits_0^\infty |I^{+}_{\eta,\alpha}f-f(z)|^p\,dz \to 0, \qquad \int\limits_0^\infty |K^{-}_{\zeta,\alpha}f-f(z)|^p\,dz \to 0$$

for $\quad f(z) \in L_p(0,\infty), \qquad \Re(\eta) > -1/p', \qquad \Re(\zeta) > -1/p$

$$(1 \leqslant p < \infty;\ 1/p+1/p' = 1),$$

when $\alpha \to 0$ while $|\arg\alpha| \leqslant \Theta < \frac{1}{2}\pi$, as will be shown in another paper.* This form of the operators was suggested to me by A. Erdélyi, who discovered the importance of them for the Hankel transform; this application of the operators, the starting-point of the present paper, will be treated in a joint paper by Erdélyi and myself.

Comparing my notation with that of Love-Young, we have formally

$$f^{+}_{\alpha}(0,z) = \{\Gamma(\alpha)\}^{-1}\int\limits_0^z (z-t)^{\alpha-1}f(t)\,dt = z^{\eta+\alpha}I^{+}_{\eta,\alpha}\{t^{-\eta}f(t)\}, \tag{1.5}$$

$$f^{-}_{\alpha}(z,\infty) = \{\Gamma(\alpha)\}^{-1}\int\limits_z^\infty (t-z)^{\alpha-1}f(t)\,dt = z^{-\eta}K^{-}_{\eta,\alpha}\{t^{\eta+\alpha}f(t)\}, \tag{1.6}$$

and so, for instance,

$$f^{+}_{\alpha}(0,z) = z^{\alpha}I^{+}_{0,\alpha}f = I^{\pm}_{-\alpha,\alpha}\{t^{\alpha}f(t)\}.$$

Generally speaking, most of our results apply to f^{+}_{α} when

$$0 < \Re(\alpha) < \infty, \quad f \in L_p$$

for $1 < p \leqslant \infty$ only, not for $p = 1$, and to f^{-}_{α} for $1 \leqslant p < \infty$, $0 < \Re(\alpha) < 1/p$, as a consequence of our dealing with the infinite interval.

In this paper I shall also discuss fractional derivatives. The main tool that I shall employ is the Mellin transform.† I do not make

* Cf. HL. 2, 582, and J. D. Tamarkin, *Annals of Math.* (2) 31 (1930), 219–28, Theorem 1.

† Dr. A. Erdélyi, has suggested to me the application of the Mellin transform to the operators $I_{\eta,\alpha}$, $K_{\eta,\alpha}$ as Hankel transforms.

any hypothesis of the nature of periodicity or almost periodicity, and I do not refer at all to the work of Hardy-Littlewood or Weyl and their much deeper theory which makes use of Lipschitz conditions. Corresponding to Weyl's* definition, we define the differentiating operators

$$f(t) = t^{-\eta}\frac{d^\alpha}{dt^\alpha}\{t^{\eta+\alpha}g^+_{\eta,\alpha}(t)\}, \tag{1.7}$$

$$f(t) = t^{\eta+\alpha}\frac{d^\alpha}{d(-t)^\alpha}\{t^{-\eta}h^-_{\eta,\alpha}(t)\}, \tag{1.8}$$

$$f(t) = t^{-\eta}\frac{d^\alpha}{d(-t)^\alpha}\{t^{\eta+\alpha}g^-_{\eta,\alpha}(t)\}, \tag{1.9}$$

$$f(t) = t^{\eta+\alpha}\frac{d^\alpha}{dt^\alpha}\{t^{-\eta}h^+_{\eta,\alpha}(t)\} \tag{1.10}$$

by the property of $f(t)$ of being the unique solution of (1.1), (1.2), (1.3), or (1.4) respectively, when such a solution exists.

The requirement of a unique solution belonging to

$$L_p(0,\infty) \quad (1 \leqslant p \leqslant \infty)$$

by Theorem 7 gives a condition which is necessary in the case $1 \leqslant p < 2$; necessary and sufficient when $p = 2$; and sufficient but not necessary when $2 < p \leqslant \infty$.

We shall use the following notations: $L_p(a,b)$ $(-\infty \leqslant a < b \leqslant \infty)$ denotes the space of all complex-valued functions $f(t)$ whose pth power is integrable over (a,b), with the norm

$$|f|_p = \left(\int_a^b |f(t)|^p\,dt\right)^{1/p} \quad (1 \leqslant p < \infty);$$

$L_\infty(a,b)$ denotes the space of all measurable functions $f(t)$ which are essentially bounded in (a,b), with the norm

$$|f|_\infty = \underset{a \leqslant t \leqslant b}{\text{essential upper bound}}\,|f(t)|.$$

We take $\mathfrak{R}(\alpha)$ as positive and finite throughout this paper, and denote constants depending on the given parameters by the single symbol K.

* Loc. cit., § 1. Of course this definition is narrower in some sense than that given by Hardy-Littlewood and generalized by Tamarkin.

2. An inequality

THEOREM 1. *Let $f(x)$ and $\phi(x)$ be measurable over* $(0,\infty)$, *let* $\lambda+\rho+\sigma = 2-1/p-1/r$ *and* $\Re(\sigma) < 1-1/p = 1/p'$, *then*

$$\left|\int_0^\infty dz \int_0^z \frac{f(t)\phi(z)\,dt}{t^\sigma z^\rho (z-t)^\lambda}\right| \leqslant K|f|_p|\phi|_r \quad (K = K(p,r,\lambda,\rho,\sigma)) \tag{2.1}$$

*under the following alternative conditions:**

(i) $1 \leqslant p \leqslant \infty,\quad 1/p+1/r = 1,\quad -\infty < \Re(\lambda) < 1;$

(ii) $p > 1,\ r > 1,\ 1/p+1/r > 1,\ 1-1/p \leqslant \Re(\lambda) \leqslant 2-1/p-1/r;$

(iii) $1 \leqslant p \leqslant \infty,\ 1 \leqslant r \leqslant \infty,\ 1/p+1/r \geqslant 1,\ -\infty < \Re(\lambda) < 1-1/p,$

and when $p = 1$ we can also take $\Re(\lambda) = 1-1/p = 0$.

Plainly (2.1) has a meaning only when $f \in L_p$, $\phi \in L_r$. Without loss of generality we may suppose $f(t) \geqslant 0$, $\phi(z) \geqslant 0$ and λ, ρ, σ to be real.

To prove 1 (i) we proceed by a method due to Hardy-Littlewood-Pólya, employing a well-known theorem,† due to them and to Schur, in a slightly generalized form.

LEMMA 1. *Let $1 \leqslant p \leqslant \infty$, let $H(x,y)$ be homogeneous of degree -1, let $H(x,1)$ and $f(x)$ be measurable over* $(0,\infty)$, *and let*

$$K = K(H,p) = \int_0^\infty |H(x,1)|x^{-1/p}\,dx = \int_0^\infty |H(1,y)|y^{-1/p'}\,dy,$$

then

$$\left|\int_0^\infty H(t,z)f(t)\,dt\right|_p \leqslant K|f(z)|_p \quad (1 < p \leqslant \infty), \tag{2.2}$$

$$\left|\int_0^\infty |H(t,z)f(t)|\,dt\right|_{p=1} = K|f(z)|_{p=1}. \tag{2.21}$$

Now, in consequence of Hölder's inequality and its converse, (2.1) is equivalent to

$$\left|\int_0^z \frac{f(t)\,dt}{z^\rho t^\sigma (z-t)^\lambda}\right|_{r'} \leqslant K|f(z)|_p \quad \left(r' = \frac{r}{r-1}\right). \tag{2.3}$$

* We also note the alternative condition (iv): $1 < p < \infty$, $1/p+1/r \geqslant 1$, $-\infty < \Re(\lambda) < 1$, $|f(t)|$ is a decreasing function. The proof follows the lines of that given in *Inequalities*, p. 289.

† I. Schur, *J. für Math.* 140 (1911), 1–28; HLP. 1; *Inequalities*, Theorem 319, cf. Theorem 329. In Lemma 1, $f(x)$ may be complex-valued. We can also show that for $1 < p \leqslant \infty$ or $p = 1$

$$\left|z^{1-2/p}\int_0^\infty |H(z,t)f(t)|\,dt\right|_p \leqslant K|z^{1-2/p}f(z)|_p \quad \text{or} \quad = K|z^{-1}f(z)|_1$$

respectively.

Taking $$H(t,z) = \begin{cases} t^{-\sigma}z^{-\rho}(z-t)^{-\lambda} & (0 \leqslant t \leqslant z), \\ 0 & (t > z) \end{cases}$$

which is a homogeneous function of t and z of degree -1, with the conditions of theorem 1 (i) and $r' = p$, we immediately get (2.3) for

$$1/p+1/r = 1, \qquad K = \int_0^1 (1-t)^{-\lambda}t^{-\sigma-1/p}\,dt = \frac{\Gamma(1-\lambda)\Gamma(1/p'-\sigma)}{\Gamma(1+1/p'-\lambda-\sigma)}.$$

Now I shall prove 1 (ii), even when $0 < \Re(\lambda) \leqslant 2-1/p-1/r$. The case $0 < \Re(\lambda) < 1/p$ is covered also by the better results of 1 (iii); the case* $\Re(\lambda) < 1$, $1/p+1/r = 1$ is covered by 1 (i). We make use of the Hardy-Littlewood-Pólya inequality

$$\int_0^\infty \int_0^\infty \frac{f(t)\phi(z)\,dtdz}{t^\sigma z^\rho |z-t|^\lambda} \leqslant K|f|_p|\phi|_r, \tag{2.4}$$

where

$$f(t) \geqslant 0, \qquad \phi(z) \geqslant 0, \qquad r > 1, \qquad p > 1, \qquad 1/p+1/r > 1,$$

$$\rho < 1/r', \qquad \sigma < 1/p',$$

$$\rho+\sigma \geqslant 0, \qquad \lambda+\rho+\sigma = 2-1/p-1/r. \tag{2.41}$$

Hence *a fortiori* (2.1) is true under the conditions (2.41).

When we take $0 < \delta < 1$, $\sigma_0 = 1/p'-\delta\lambda$, $\rho_0 = 1/r'-\lambda(1-\delta)$, then plainly

$$\sigma_0 < 1/p', \qquad \lambda+\rho_0+\sigma_0 = 2-1/p-1/r, \qquad \rho_0 < 1/r'. \tag{2.42}$$

Now let p, r, ρ, σ satisfy the conditions of 1 (ii) and let

$$0 < \lambda \leqslant 2-1/p-1/r,$$

then $\sigma < \sigma_0$ when δ is sufficiently small, and, in consequence of (2.42), we have

$$\rho+\sigma = \rho_0+\sigma_0, \qquad \rho_0+\sigma_0 \geqslant 0. \tag{2.43}$$

Therefore (2.1) holds when we replace ρ by ρ_0, σ by σ_0, since p, r, ρ_0, σ_0 satisfy (2.41). But this integral majorizes the (ρ,σ) integral, for we have $0 \leqslant t \leqslant z$, $t^{\rho_0}z^{\sigma_0} = t^\rho z^\sigma(tz^{-1})^{\sigma-\sigma_0} \geqslant t^\rho z^\sigma$, and so 1 (ii) is proved. When we interchange ρ, p, f, t with σ, r, ϕ, z respectively, and add the result to (2.1), we see that Theorem 1 implies the Hardy-Littlewood-Pólya inequality.

* Therefore we did not mention this case (vide HL. 2) in Theorem 1 (ii).

In proving 1 (iii) we may suppose $p < \infty$, since $p = \infty$ makes $r = \infty$, and this case is covered by 1 (i). Let

$$V(z) = z^{-\rho} \int_0^z \frac{f(t)\,dt;}{t^\sigma(z-t)^\lambda}; \tag{2.5}$$

then, by Hölder's theorem,

$$V(z)z^{1/r'} \leqslant \{\Gamma(1-\sigma p')\Gamma(1-\lambda p')/\Gamma(2-\sigma p'-\lambda p')\}^{1/p'}|f|_p = K|f|_p, \tag{2.6}$$

when $p > 1$. When $p = 1$, we have $\sigma < 0$, $\lambda < 0$,

$$V(z)z^{1/r'} \leqslant |f|_1 \max_{0 \leqslant t \leqslant z} \left(\frac{z}{t}\right)^\sigma \left(1-\frac{t}{z}\right)^{-\lambda} \leqslant |f|_1. \tag{2.61}$$

When $r' = \infty$ we immediately get (2.3). Now let $1 \leqslant p \leqslant r' < \infty$. Then

$$\int_0^\infty \{V(z)\}^{r'}\,dz = \int_0^\infty \{z^{1/r'}V\}^{r'-p}\{z^{1/r'-1/p}V\}^p\,dz \leqslant (K|f|_p)^{r'-p}|W|_p^p, \tag{2.7}$$

where $\displaystyle W(z) = \int_0^z f(t)z^{-\rho_1}t^{-\sigma}(z-t)^{-\lambda}\,dt, \qquad \rho_1 = \rho+1/p-1/r',$

and $\lambda+\rho_1+\sigma = \lambda+\rho+\sigma+1/p-1/r' = 1 = 2-1/p-1/p'$. Hence, taking $r = p'$, by 1 (i) we have $|W(z)|_p \leqslant K|f|_p$, and from (2.7) the result $|V(z)|_{r'} \leqslant K|f|_p$ easily follows for any r satisfying (iii).

When $p = 1$ and $\lambda = 1-1/p = 0$, we have $\sigma < 0$, $\rho+\sigma = 1-1/r$; then

$$z^{1/r'}V(z) = z^\sigma \int_0^z t^{-\sigma}f(t)\,dt = \int_0^z \left(\frac{z}{t}\right)^\sigma f(t)\,dt \leqslant |f|_1, \tag{2.8}$$

and so $|V(z)|_{r'} \leqslant |f|_1$ for $r = 1$. When $r > 1$, then

$$\int_0^\infty V^{r'}\,dz$$

$$= \int_0^\infty (z^{1/r'}V)^{r'-1}z^{-1/r}V\,dz \leqslant |f|_1^{r'-1} \int_0^\infty dz\, z^{\sigma-1} \int_0^z t^{-\sigma}f(t)\,dt = |\sigma|^{-1}|f|_1^{r'},$$

and so $|V|_{r'} \leqslant |\sigma|^{1/r-1}|f|_1$. Hence (2.3) is true.

3. Some properties of the operators $I_{\eta,\alpha}, K_{\eta,\alpha}$

We have

$$I^-_{\eta,\alpha}f = K^-_{-\eta-\alpha,\alpha}f, \qquad K^+_{\eta,\alpha}f = I^+_{-\eta-\alpha,\alpha}f, \tag{3.1}$$

$$I^+_{\eta,\alpha}\{t^{-1}f(t^{-1})\} = z^{-1}h^-_{\eta,\alpha}(z^{-1}), \qquad K^-_{\eta,\alpha}\{t^{-1}f(t^{-1})\} = z^{-1}g^+_{\eta,\alpha}(z^{-1}), \tag{3.2}$$

$$z^a I^+_{\eta,\alpha}f = I^+_{\eta-a,\alpha}\{t^a f(t)\}, \qquad z^a K^-_{\eta,\alpha}f = K^-_{\eta+a,\alpha}\{t^a f(t)\} \tag{3.3}$$

at all points z for which either side of the equation exists.

The proofs of (3.1) and (3.3) are trivial, the proof of (3.2) is not difficult. Furthermore, we have

THEOREM 2. *Let $f(t)$ belong to $L_p(0,\infty)$ and let $\Re(\zeta) > -1/p'$, $\Re(\eta) > -1/p$, then the functions $g^+_{\zeta,\alpha}(z) = I^+_{\zeta,\alpha}f$, $h^-_{\eta,\alpha}(z) = K^-_{\eta,\alpha}f$ exist almost everywhere in* $(0,\infty)$. *Moreover,*

$$|z^{1/p-1/q}g^+_{\zeta,\alpha}(z)|_q \leqslant K|f|_p, \qquad (K = K(p,q,\alpha,\eta,\zeta)) \qquad (3.4)$$

$$|z^{1/p-1/q}h^-_{\eta,\alpha}(z)|_q \leqslant K|f|_p \qquad (3.5)$$

under the following additional alternative hypotheses

(i) $1 \leqslant p \leqslant \infty, \quad q = p;$

(ii) $1 < p < q < \infty, \quad 1/p-1/q \leqslant \Re(\alpha) \leqslant 1/p;$

(iii) $1 \leqslant p \leqslant q \leqslant \infty, \quad \Re(\alpha) > 1/p;$ *when* $p = 1 \leqslant q \leqslant \infty,$
$$\Re(\alpha) \geqslant 1/p = 1.$$

The results 2 (i) and 2 (ii) on $I^+_{\eta,\alpha}$ are in substance due to Hardy-Littlewood,* so is 2 (i) for $K^-_{0,\alpha}$. Since the corresponding results on I^- and K^+ easily follow from those on K^- and I^+ by (3.1), we do not discuss I^- or K^+ further.

In (2.3) we put $\lambda = 1-\alpha$, $\sigma = -\zeta$, $\rho = \alpha+\zeta-1/p+1/q$, $r = q'$, then plainly $\{\Gamma(\alpha)\}^{-1}V(z) = z^{1/p-1/q}I^+_{\zeta,\alpha}f$, and the conditions (i), (ii), (iii) of Theorem 2 are respectively equivalent to (i), (ii), (iii) of Theorem 1; therefore the results on $I^+_{\zeta,\alpha}$ are an immediate consequence of Theorem 1: the existence of $I^+_{\zeta,\alpha}$ is now a consequence of (3.4).

Putting $\zeta = \eta-a$ and $f(t) = t^a\phi(t)$, we have $\Re(\eta) > \Re(a)-1/p'$,

$$z^{1/p-1/q}I^+_{\eta-a,\alpha}\{t^a\phi(t)\} = z^{1/p-1/q+a}I^+_{\eta,\alpha}\{\phi(t)\}$$

in consequence of (3.3), and therefore, by (3.4),

$$|z^{1/p-1/q+a}I^+_{\eta,\alpha}\phi|_q \leqslant K|t^a\phi(t)|_p \quad (\Re(\eta) > \Re(a)-1/p') \qquad (3.41)$$

under the alternative conditions (i), (ii), (iii) of Theorem 2; (3.41) is slightly more general than (3.4). Putting $a = 1-2/p$, we get

$$|z^{1/p'-1/q}I^+_{\eta,\alpha}\phi|_q \leqslant K|t^{1-2/p}\phi(t)|_p \quad (\Re(\eta) > -1/p). \qquad (3.42)$$

Taking $\phi(t) = t^{-1}\psi(t^{-1})$ and using (3.2), for $\Re(\eta) > -1/p$, we have

$$|z^{1/p-1/q}K^-_{\eta,\alpha}\psi|_q = |z^{1/p'-1/q}I^+_{\eta,\alpha}\phi|_q \leqslant K|t^{1-2/p}\phi|_p = K|\psi|_p.$$

We have thus proved (3.5) and, taking $\psi(t) = t^bf(t)$, $\zeta = \eta-b$, the more general inequality

$$|z^{1/p-1/q+b}K^-_{\zeta,\alpha}f|_q \leqslant K|t^bf(t)|_p \quad (\Re(\zeta) > -1/p-\Re(b)). \qquad (3.51)$$

* HL. 2, Theorems 7–11, and *Inequalities*, Theorem 329.

I shall now give some formulae on fractional integration by parts and repeated fractional integration.

THEOREM 3. *Let a, b be any numbers such that $a+b = 1/q-1/p$ and let $t^a f(t) \in L_p(0,\infty)$, $t^b\phi(t) \in L_{p'}(0,\infty)$, then*

$$\int_0^\infty dz\,\phi(z)I^+_{\eta,\alpha}f = \int_0^\infty dz\,f(z)K^-_{\eta,\alpha}\phi \quad (\Re(\eta) > \Re(a)-1/p'), \tag{3.61}$$

$$\int_0^\infty dz\,\phi(z)I^-_{\eta,\alpha}f = \int_0^\infty dz\,f(z)K^+_{\eta,\alpha}\phi \quad (\Re(\eta+a) > \Re(a)+1/p') \tag{3.62}$$

under the following alternative conditions:

(i) $1 \leqslant p \leqslant \infty, \qquad q = p;$

(ii) $1 < p < \infty, \qquad 1/p-1/q \leqslant \Re(\alpha) \leqslant 1/p;$

(iii) $1 \leqslant p \leqslant q \leqslant \infty, \qquad \Re(\alpha) > 1/p;$ *when* $p = 1 \leqslant q \leqslant \infty,$

$$\Re(\alpha) \geqslant 1/p = 1.$$

Proof. In consequence of Theorem 2 and (3.4), the function $\chi(z) = |z^{1/p-1/q+a}|\,I^+_{\Re(\eta),\Re(\alpha)}\{|f(t)|\}$ belongs to $L_q(0,\infty)$, and by

$$1/p-1/q+a = -b,$$

$$|\phi(z)|\,I^+_{\Re(\eta),\Re(\alpha)}\{|f(t)|\} = |z^b\phi(z)|\chi(z) \in L_1(0,\infty),$$

we may therefore interchange the integrations

$$\int_0^\infty dz\,\phi(z)I^+_{\eta,\alpha}f = \frac{1}{\Gamma(\alpha)}\int_0^\infty dz\,\phi(z)z^{-\eta-\alpha}\int_0^z (z-t)^{\alpha-1}t^\eta f(t)\,dt$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^\infty dt\,t^\eta f(t)\int_t^\infty (z-t)^{\alpha-1}z^{-\eta-\alpha}\phi(z)\,dz = \int_0^\infty dt\,f(t)K^-_{\eta,\alpha}\phi.$$

Hence (3.61) is true. Taking $a = \eta = 0$, $z^{-\alpha}\phi(z) = g(z)$, $f(z) = g(z) = 0$ for $z > c$, where c is any positive number, then, in consequence of (1.5) and (1.6), using the Love-Young notation,

$$\int_0^c g(z)f^+_\alpha(z)\,dz = \int_0^c \phi(z)I^+_{0,\alpha}f\,dz = \int_0^c f(z)K^-_{0,\alpha}\{t^\alpha g(t)\}\,dz = \int_0^c f(z)g^-_\alpha(z)\,dz$$

when $\qquad f \in L_p(0,c), \qquad p > 1, \qquad z^{\alpha+1/q-1/p}g(z) \in L_{q'}(0,c)$

and one of the conditions (i), (ii), (iii) is satisfied. This result is in some respects better and in some respects worse than Theorem C

of Love-Young: incidentally, it is easy to show that, for $\Re(\alpha) \geqslant 1$, the Love-Young equation holds for any $p \geqslant 1, r \geqslant 1$ when

$$f(z) \in L_p(c_1, c_2), \qquad g(z) \in L_r(c_1, c_2) \quad (-\infty < c_1 < c_2 < \infty),$$

and it is not difficult to give sets of sufficient conditions for negative values of $\Re(\alpha)$.

From Theorem 3 and from (3.2) we get

LEMMA 2. *Let $1 \leqslant p \leqslant \infty$ and $\Re(\eta) > -1/p'$, $f(t) \in L_p$, $t^{1-2/p}\phi(t) \in L_{p'}$, then*

$$\int_0^\infty \frac{dz}{z}\phi(z^{-1}) I^+_{\eta,\alpha} f = \int_0^\infty \frac{dz}{z} f(z^{-1}) I^+_{\eta,\alpha}\phi, \tag{3.63}$$

and the corresponding equations are valid for $K^-_{\eta,\alpha}$, $I^-_{\eta,\alpha}$, $K^+_{\eta,\alpha}$ when $\Re(\eta) > -1/p$ or $< 1/p - \Re(\alpha)$ or $< 1/p' - \Re(\alpha)$ respectively.

By using Theorem 3 to its full extent we can also get some more general conditions.

The following theorem* concerns repeated fractional integration.

THEOREM 4. *Let $\Re(\lambda) > 0$, $\Re(\mu) > 0$, let $f(t) \in L_p$ or $t^{1-2/p} f(t) \in L_p$ $(1 \leqslant p \leqslant \infty)$, then, for $\Re(\eta) > -1/p'$ or $\Re(\eta) > -1/p$ respectively,*

$$I^+_{\eta,\lambda}(I^+_{\eta+\lambda,\mu} f) = I^+_{\eta,\lambda+\mu} f = I^+_{\eta+\lambda,\mu}(I^+_{\eta,\lambda} f) = I^+_{\eta,\mu}(I^+_{\eta-\mu,\lambda}) = I^+_{\eta+\mu,\lambda}(I^+_{\eta,\mu} f). \tag{3.7}$$

When $\Re(\eta) > -1/p$ or $> -1/p'$, the corresponding equation holds for $K^-_{\eta,\alpha}$.

We have

$$I^+_{\eta,\lambda}(I^+_{\eta+\lambda,\mu} f) = \frac{z^{-\eta-\lambda}}{\Gamma(\lambda)\Gamma(\mu)} \int_0^z (z-t)^{\lambda-1} t^{-\lambda-\mu}\, dt \int_0^t (t-x)^{\mu-1} x^{\eta+\lambda} f(x)\, dx$$

$$= \frac{z^{-\eta-\lambda}}{\Gamma(\lambda)\Gamma(\mu)} \int_0^z x^{\eta+\lambda} f(x) W(x)\, dx, \tag{3.71}$$

where

$$W(x) = \int_x^z (z-t)^{\lambda-1}(t-x)^{\mu-1} t^{-\lambda-\mu}\, dt = \frac{\Gamma(\lambda)\Gamma(\mu)(z-x)^{\lambda+\mu-1}}{\Gamma(\lambda+\mu) x^\lambda z^\mu}, \tag{3.72}$$

the interchanging of the integrations being justified as in the proof

* For the results on $f^+_\alpha(0, z)$ and $f^-_\alpha(z, \infty)$ cf. Weyl, Love, Tamarkin, loc. cit., and E. Hille, *Annals of Math.* 40 (1939), 1–47; 4.4.

of Theorem 3, and so the first part of Theorem 4 follows from (3.4) and (3.42) when we take $q = p$; by similar reasoning we get

$$I^{+}_{\eta+\lambda,\mu}(I^{+}_{\eta,\lambda}f) = I^{+}_{\eta,\lambda+\mu}f,$$

and the remaining assertions by interchanging λ and μ.

As an application of (3.7), take n to be an integer, $0 < \Re(\lambda) < n$, $\mu = n-\lambda$, and we easily get

$$\frac{1}{\Gamma(n)}\int\limits_0^z (z-t)^{n-1}t^{\eta}f(t)\,dt = \frac{1}{\Gamma(n-\lambda)}\int\limits_0^z (z-t)^{n-\lambda-1}t^{\eta+\lambda}g^{+}_{\eta,\lambda}(t)\,dt, \tag{3.73}$$

and by the definition (1.7), writing g instead of $g^{+}_{\eta,\alpha}$,

$$z^{\eta}f(z) = \frac{d^{\lambda}}{dz^{\lambda}}\{z^{\eta+\lambda}g(z)\} = \frac{1}{\Gamma(n-\lambda)}\frac{d^n}{dz^n}\int\limits_0^z (z-t)^{n-\lambda-1}t^{\eta+\lambda}g(t)\,dt,$$

if the right-hand term of (3.73) is representable as the n-fold integral with origin 0 of a function $\phi(z)$ such that $z^{-\eta}\phi(z) = f(z) \in L_p$ or $z^{1-2/p-\eta}\phi(z) = z^{1-2/p}f(z) \in L_p$ for some $p \geqslant 1$ where $\Re(\eta) > -1/p'$ or $\Re(\eta) > -1/p$. This is the well-known solution of Abel's integral equation.

4. Mellin transforms

Before dealing with fractional derivatives we have to discuss the Mellin transforms* of $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$.

It is well known that the Mellin transform $f_1(\tau) = \mathfrak{M}f$ exists and belongs to $L_{p'}(-\infty,\infty)$ when $1 \leqslant p \leqslant 2$ and $f(x)$ belongs to $L_p(0,\infty)$,

$$f_1(\tau) = \int\limits_0^{\infty} f(x)x^{s-1}\,dx \quad (p = 1),$$

$$f_1(\tau) = \underset{N\to\infty}{\text{l.i.m.}}\int\limits_{1/N}^{N} f(x)x^{s-1}\,dx \quad \begin{pmatrix}\text{index } p',\\ 1 < p \leqslant 2\end{pmatrix}, \tag{4.1}$$

* The theorems on Mellin transforms are an easy consequence (cf. E. C. Titchmarsh, *The Theory of Functions* (Oxford 1932), 443) of those on Fourier transforms (E. C. Titchmarsh, *Proc. London Math. Soc.* (2) 23 (1926), 279–89). Evidently (4.1) is equivalent to $f_1(\tau) = \int \chi(y)e^{-iy\tau}\,dy$; where

$$\chi(y) = f(e^{-y})\exp(-y/p) \in L_p(-\infty,\infty),$$

and (4.2) to $g(y) = (2\pi)^{-1}\int \phi(\tau)e^{-iy\tau}\,d\tau$, where

$$g(y) = \psi(y)\exp(-y/q') \in L_{q'}(-\infty,\infty).$$

Hence the operator $\mathfrak{M}$ is continuous for $1 \leqslant p \leqslant 2$, but discontinuous for $p > 2$. For Hille-Tamarkin's theorem vide *Bull. American Math. Soc.* 39 (1933), 768–74.

where $s = 1/p+i\tau$. When $\phi(\tau)$ belongs to $L_q(-\infty,\infty)$ and $1 \leqslant q \leqslant 2$, then the function

$$\psi(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \phi(\tau)x^{-i\tau}\,d\tau \quad (q=1),$$

$$\psi(x) = \frac{1}{2\pi}\operatorname*{l.i.m.}_{N\to\infty}\int\limits_{-N}^{N} \phi(\tau)x^{-1/q'-i\tau}\,d\tau \quad \begin{pmatrix}\text{index } q',\\ 1<q\leqslant 2\end{pmatrix} \tag{4.2}$$

belongs to $L_{q'}(0,\infty)$, and we shall denote it by $\psi = \mathfrak{M}^{-1}\phi$.

Now let us denote by $\mathfrak{M}_p$ $(2 < p \leqslant \infty)$ the set of all functions $f(x)$ which have a Mellin transform $\phi(\tau)$ in the well-defined sense that they are representable in the form $\mathfrak{M}^{-1}\phi$, where $\phi(\tau) \in L_{p'}(-\infty,\infty)$, $1 \leqslant p' = p/(p-1) < 2$. It is well known that $\mathfrak{M}_p$ is a sub-space of $L_p(0,\infty)$ and different from L_p, and that, in consequence of Hille-Tamarkin's theorem, for $2 < p < \infty$

$$\phi(\tau) = \operatorname*{l.i.m.}_{N\to\infty}\int\limits_{1/N}^{N} f(x)x^{s-1}\,dx \quad \begin{pmatrix}\text{index } p',\\ s = 1/p+i\tau\end{pmatrix}. \tag{4.11}$$

When $x^{1-2/p}f(x)$ belongs to $L_p(0,\infty)$, then $F(x) = x^{-1}f(x^{-1})$ also belongs to L_p and $f_1(\tau) = F_1(-\tau)$, and so we confine ourselves to the case $f(x) \in L_p(0,\infty)$.

THEOREM 5 (a). *Let $1 \leqslant p \leqslant 2$ and $f \in L_p(0,\infty)$, then*

$$g_1^+(\tau) = \mathfrak{M}I_{\eta,\alpha}^+ f = \Gamma(\eta+1/p'-i\tau)\{\Gamma(\eta+\alpha+1/p'-i\tau)\}^{-1}f_1(\tau)$$
$$(\Re(\eta) > -1/p'), \tag{4.3}$$

$$h_1^-(\tau) = \mathfrak{M}K_{\eta,\alpha}^- f = \Gamma(\eta+1/p+i\tau)\{\Gamma(\eta+\alpha+1/p+i\tau)\}^{-1}f_1(\tau)$$
$$(\Re(\eta) > -1/p). \tag{4.4}$$

THEOREM 5 (b). *Let $2 < p \leqslant \infty$ and let $f(t) \in \mathfrak{M}_p$, then $I_{\eta,\alpha}^+ f$ and $K_{\eta,\alpha}^- f$ also belong to $\mathfrak{M}_p$, and (4.3) and (4.4) are also valid.*

Proof of 5 (a). Let $1 \leqslant p \leqslant 2$ and $1 < a < \infty$, and let $f(x,a) = f(x)$ for $1/a < x < a$, $f(x) = 0$ otherwise. Then

$$\int\limits_0^z dz\, z^{-1/p'+i\tau} I_{\eta,\alpha}^+\{f(t,a)\} = \frac{1}{\Gamma(\alpha)}\int\limits_0^\infty f(t,a)t^\eta\,dt \int\limits_t^\infty z^{-1/p'-\eta-\alpha+i\tau}(z-t)^{\alpha-1}\,dz$$

$$= \frac{\Gamma(\eta+1/p'-i\tau)}{\Gamma(\eta+\alpha+1/p'-i\tau)}\int\limits_{1/a}^{a} t^{-1/p'+i\tau}f(t)\,dt. \tag{4.5}$$

The interchanging of the integrations and so the existence of the left-hand integral is justified by absolute convergence, for

$$|t^\eta| \int\limits_t^\infty |z^{-1/p'-\eta-\alpha+i\tau}(z-t)^{\alpha-1}|\, dz = \frac{\Gamma\{\Re(\alpha)\}\Gamma\{\Re(\eta)+1/p'\}}{\Gamma\{\Re(\eta+\alpha)+1/p'\}} t^{-1/p'} < Kt^{-1/p'}.$$

Hence, by a well-known theorem on mean convergence, the left-hand side of (4.5) is equal to $\mathfrak{M} I^+_{\eta,\alpha}\{f(t,a)\}$; the right-hand side is equal to $\omega(\tau)\mathfrak{M}\{f(t,a)\}$, where

$$\omega(\tau) = \Gamma(\eta+1/p'-i\tau)/\Gamma(\eta+\alpha+1/p'-i\tau), \tag{4.6}$$

and so
$$\mathfrak{M} I^+_{\eta,\alpha}\{f(t,a)\} = \omega(\tau)\mathfrak{M} f\{t,a\}. \tag{4.7}$$

Now $|f(t)-f(t,a)|_p \to 0$ $(a \to \infty)$; since the operator $\mathfrak{M}$ is known to be continuous for $1 \leqslant p \leqslant 2$ and the function $\omega(\tau)$ is bounded in $(-\infty,\infty)$, we have in $L_{p'}(-\infty,\infty)$

$$|\omega(\tau)\mathfrak{M} f(t)-\omega(\tau)\mathfrak{M} f(t,a)|_p \leqslant K|\mathfrak{M}\{f(t)-f(t,a)\}|_{p'} \to 0. \tag{4.8}$$

Also

$$|\mathfrak{M} I^+_{\eta,\alpha} f(t)-\mathfrak{M} I^+_{\eta,\alpha} f(t,a)|_{p'} = |\mathfrak{M} I^+_{\eta,\alpha}\{f(t)-f(t,a)\}|_{p'} \to 0, \tag{4.9}$$

since, by Theorem 2 (i), the operator $I^+_{\eta,\alpha}$ is continuous in L_p. Therefore we have

$$\mathfrak{M} I^+_{\eta,\alpha} f = \omega(\tau)\mathfrak{M} f. \tag{4$'$.3}$$

Similarly we prove (4.4).

Proof of 5 (*b*). Let $2 < p \leqslant \infty$ and let $f \in \mathfrak{M}_p$, then, by (4.2),

$$f(x) = f(x;\infty) \quad (p = \infty)$$

or
$$f(x) = \text{l.i.m.}\, f(x;N) \quad (\text{index } p,\ p < \infty,\ N \to \infty)$$

when $f_1(\tau) \in L_{p'}(-\infty,\infty)$ is the Mellin transform of $f(x)$ and when

$$f(x;N) = \frac{1}{2\pi}\int\limits_{-N}^{N} f_1(\tau)x^{-1/p-i\tau}\, d\tau.$$

Hence

$$g^+_{\eta,\alpha}(z;N) = I^+_{\eta,\alpha}\{f(t;N)\} = \frac{z^{-\eta-\alpha}}{2\pi\Gamma(\alpha)}\int\limits_0^z (z-t)^{\alpha-1}t^\eta\, dt \int\limits_{-N}^{N} f_1(\tau)t^{-1/p-i\tau}\, d\tau.$$

$$= \frac{1}{2\pi}\int\limits_{-N}^{N} d\tau\, f_1(\tau)\frac{z^{-\eta-\alpha}}{\Gamma(\alpha)}\int\limits_0^z (z-t)^{\alpha-1}t^{\eta-1/p-i\tau}\, dt,$$

the interchanging of the integrations being justified by the absolute convergence of the latter repeated integral, and

$$g^{+}_{\eta,\alpha}(z;N) = \frac{1}{2\pi}\int\limits_{-N}^{N} d\tau\, z^{-1/p-i\tau}\left\{f_1(\tau)\frac{\Gamma(\eta+1/p'-i\tau)}{\Gamma(\eta+\alpha+1/p'-i\tau)}\right\}.$$

We denote the term in the brackets $\{\}$ by $\phi(\tau)$; then plainly

$$\phi(\tau) \in L_{p'}(-\infty,\infty).$$

Hence $\mathfrak{M}^{-1}\phi$ exists*, $|g^{+}_{\eta,\alpha}(z;N)-\mathfrak{M}^{-1}\phi|_p \to 0$ $(N\to\infty)$, and, from

$$|g^{+}_{\eta,\alpha}(z;N)-g^{+}_{\eta,\alpha}(z)|_p = |I^{+}_{\eta,\alpha}\{f(t,N)-f(t)\}|_p \leqslant K|f(t,N)-f(t)|_p \to 0,$$

we get $g^{+}_{\eta,\alpha}(z) = \mathfrak{M}^{-1}\phi$, $\phi = g_1^{+}(\tau)$, and (4.3). Similarly we prove the second assertion of 5 (b).

By the same argument, using the Hardy-Littlewood-Pólya-Schur theorem, we obtain the more general result.

THEOREM 5. *Let* $f \in L_p(0,\infty)$ $(1 \leqslant p \leqslant 2)$ *or* $f \in \mathfrak{M}_p$ $(2 < p \leqslant \infty)$, *let* $K(x,y)$ *be homogeneous of degree* -1 *and* $K(1,y)y^{-1/p'} \in L_1(0,\infty)$, *then*

$$\mathfrak{M}\left\{\int\limits_0^\infty K(t,y)f(t)\,dt\right\} = \int\limits_0^\infty K(1,z)z^{-1/p'+i\tau}\,dz\,\mathfrak{M}f.$$

5. Fractional derivatives

When, in the equation (1.1), $g^{+}_{\eta,\alpha}(z)$ is given and we seek a solution $f(t)$ such that the right-hand side exists almost everywhere and is equal to $g^{+}(z)$ almost everywhere in $(0,\infty)$, we see that the definition (1.7) is narrower than the ordinary one for integer values of α. For instance, taking $\alpha = 1$, $g^{+}(z)z^{\eta+\alpha}$ must be equal to an integral with lower limit zero almost everywhere in $(0,\infty)$ in order that its derivative may exist almost everywhere in accordance with our definition. First we prove the following

UNIQUENESS THEOREM.† *Let* $-\infty < a < U \leqslant \infty$, *let* $g(x)$ *be defined and finite almost everywhere in* (a,U). *If a function* $f(x)$ *exists at all, defined in* (a,U) *so that*

$$g(x) = \int\limits_a^x f(t)(x-t)^{\alpha-1}\,dt$$

almost everywhere in (a,U), *then* $f(x)$ *is uniquely determined in* (a,U) *except at a set of measure zero.*

* Cf. 4.2.

† This result was proved by Tamarkin, loc. cit., Theorem 4. Our proof is entirely different from his.

Without loss of generality we may take $a = 0$. We have only to show that, when the function

$$I(x) = \int_0^x \phi(t)(x-t)^{\alpha-1}\,dt \tag{5.1}$$

is defined in $(0, U)$ and vanishes almost everywhere in $(0, U)$, then so does $\phi(x)$. Plainly $\phi(x) \in L_1(0, A)$ for any positive $A < U$; for some B $(A < B < U)$ the integral $I(B)$ must exist in the Lebesgue sense and be finite, and so

$$\int_0^A |\phi(t)|\,dt \leqslant C\int_0^A |(B-t)^{\alpha-1}\phi(t)|\,dt \leqslant C\int_0^B |(B-t)^{\alpha-1}\phi(t)|\,dt < \infty,$$

where $C = \max\{(B-A)^{1-\Re(\alpha)}, B^{1-\Re(\alpha)}\}$. Now let A be fixed and put $x = Av$, $t = Ax'$. Then, in consequence of (5.1),

$$\int_0^v (v-x')^{\alpha-1}\phi(Ax')\,dx'$$

must exist and vanish almost everywhere for $0 \leqslant v \leqslant 1$. Multiplying by $(1-v)^n$ $(n = 0, 1, 2,...)$ and integrating we get

$$\begin{aligned} 0 &= \int_0^1 dv(1-v)^n \int_0^v (v-x)^{\alpha-1}\phi(Ax)\,dx \\ &= \int_0^1 \phi(Ax)\,dx \int_x^1 (1-v)^n(v-x)^{\alpha-1}\,dv \\ &= \int_0^1 \phi\{A(1-x)\}\,dx \int_0^x t^n(x-t)^{\alpha-1}\,dt \\ &= \frac{n!\,\Gamma(\alpha)}{\Gamma(n+\alpha+1)} \int_0^1 \phi\{A(1-x)\}x^{n+\alpha}\,dx. \end{aligned}$$

The interchanging of the integrations is justified since

$$\int t^n(x-t)^{\Re(\alpha)-1}\,dt < Kx^{n+\Re(\alpha)} \quad \text{and} \quad \phi\{A(1-x)\}x^{n+\alpha} \in L_1(0, 1).$$

Hence $$\int_0^1 \phi\{A(1-x)\}x^{n+\alpha}\,dx = 0 \quad (n = 0, 1, 2,...);$$

therefore, by Lerch's theorem, $x^\alpha\phi\{A(1-x)\} \equiv 0$ $(0 < x < 1)$, and so $\phi(x) = 0$ almost everywhere in $(0, A)$ and consequently in $(0, U)$. This completes the proof.

From this theorem we immediately establish the uniqueness of the solution f of $I^+_{\eta,\alpha} f = g$ if it exists at all, and from this result we deduce the uniqueness of the solution f of $K^-_{\eta,\alpha} f = h$ by (3.2).

Now let the fractional derivative f, defined by (1.7) or (1.8), satisfy the condition (i) that it belongs to $L_p(0,\infty)$ for some fixed p $(1 \leqslant p \leqslant \infty)$. We have*

THEOREM 6 (*a*). *When $g(z)$ has a fractional derivative of order α satisfying* (i), *then the same is true for any order β such that*

$$0 < \Re(\beta) < \Re(\alpha).$$

Proof. Let $\Re(\eta) > -1/p'$, $f \in L_p$, and let $g(z) = I^+_{\eta,\alpha} f$. Putting $\phi(z) = I^+_{\eta+\beta,\alpha-\beta} f$ we have $\phi(z) \in L_p$ in consequence of (3.4), and, by (3.7), $I^+_{\eta,\beta}\phi = I^+_{\eta,\beta}(I^+_{\eta+\beta,\alpha-\beta} f) = I^+_{\eta,\alpha} f$. Hence ϕ is the desired solution of $I^+_{\eta,\beta}\phi = g$. The same proof applies to the other operators.

Let us denote by ${}^+E^{(p)}_{\eta,\alpha}$ the set of all functions g of the form $g = I^+_{\eta,\alpha} f$ $(f \in L_p, \Re(\eta) > -1/p', 1 \leqslant p \leqslant \infty)$, and denote the corresponding sets belonging to the other operators by ${}^-F^{(p)}_{\eta,\alpha}$, ${}^-E^{(p)}_{\eta,\alpha}$, ${}^+F^{(p)}_{\eta,\alpha}$. Plainly we have ${}^+E^{(p)}_{\eta,\alpha} \subset {}^+E^{(p)}_{\eta,\beta}$ when $0 < \Re(\beta) < \Re(\alpha)$, it is probably also true that ${}^+E^{(p)}_{\eta,\alpha} = {}^-F^{(p)}_{\vartheta,\alpha} = {}^-E^{(p)}_{\rho,\alpha} = {}^+F^{(p)}_{\sigma,\alpha}$ for any numbers η, ϑ, ρ, σ such that $\Re(\eta) > -1/p'$, $\Re(\vartheta) > -1/p$, $\Re(\rho+a) < 1/p$, $\Re(\sigma+a) < 1/p'$. This certainly is the case for $p = 2$ as we shall see by Theorem 7 (*a*); when $p > 2$ and $(\mathfrak{M}_p, {}^+E^{(p)}_{\eta,\alpha})$ denotes the set of all functions of ${}^+E^{(p)}_{\eta,\alpha}$ which have a Mellin transform, then also, by Theorem 7 (*c*), $(\mathfrak{M}_p, {}^+E^{(p)}_{\eta,\alpha}) = (\mathfrak{M}_p, {}^-F^{(p)}_{\vartheta,\alpha}) = \dots$.

THEOREM 6 (*b*). *Let $1 \leqslant p < \infty$, $\Re(\eta) > -1/p'$, then the set ${}^+E^{(p)}_{\eta,\alpha}$ is dense everywhere in $L_p(0,\infty)$.*

By a well-known theorem† we have only to show that ${}^+E^{(p)}_{\eta,\alpha}$ is complete with respect to $L_{p'}(0,\infty)$. Let $\phi(x) \in L_{p'}$, let

$$\int\limits_0^\infty \bar{\phi}(x) g(x)\, dx = 0 \tag{5.2}$$

for any $g \in {}^+E^{(p)}_{\eta,\alpha}$, then we have to prove that $\phi(x) \equiv 0$ in $(0,\infty)$. Now $g(x)$ has the form $I^+_{\eta,\alpha} f$ $(f \in L_p)$, therefore by Theorem (3.1), taking $a = b = 0$, we have

$$\int\limits_0^x dx\, f(x) K^-_{\eta,\alpha} \bar{\phi} = 0$$

* Cf. Weyl, loc. cit., Satz 1, HL. 2, § 6, Tamarkin, Theorem 6.

† S. Banach, *Théorie des opérations linéaires* (Warszawa 1932), 74. The theorem is true also for the infinite interval.

for any $f \in L_p$, even when $p = \infty$; hence $K^-_{\eta,\alpha}\bar{\phi} \equiv 0$ in $(0,\infty)$, and by the uniqueness theorem, $\phi(x) = 0$ almost everywhere. This completes the proof.

The corresponding theorems hold for ${}^-F^{(p)}_{\eta,\alpha}$, ${}^-E^{(p)}_{\eta,\alpha}$, and ${}^+F^{(p)}_{\eta,\alpha}$.

6. THEOREM 7. (*a*) *Let $p = 2 = p'$; then necessary and sufficient conditions for the equation $g(z) = I^+_{\eta,\alpha}f$ $(\Re(\eta) > -1/p')$ to have a solution $f \in L_2(0,\infty)$ are that $g(z)$ belongs to $L_2(0,\infty)$ and its Mellin transform $g_1(\tau)$ satisfies the condition $\tau^\alpha g_1(\tau) \in L_2(-\infty,\infty)$.*

(*b*) *When $1 \leqslant p < 2$, the conditions $g \in L_p(0,\infty)$ and*

$$\tau^\alpha g_1(\tau) \in L_{p'}(-\infty,\infty)$$

are necessary.

(*c*) *When $2 < p \leqslant \infty$, the conditions $g(z) \in \mathfrak{M}_p$ and*

$$\tau^\alpha g_1(\tau) \in L_{p'}(-\infty,\infty)$$

are sufficient.

The same result holds for the operator $K^-_{\eta,\alpha}$ $(\Re(\eta) > -1/p)$.

Proof of 7 (*a*). When $f(t) \in L_2$, by (3.4) $g(z) \in L_2$ also. Now let a and b be real and $a \neq 0, -1, -2,\ldots$, then

$$\lim|\Gamma(a+ib+i\tau)|e^{\frac{1}{2}\pi|\tau+b|}|\tau+b|^{\frac{1}{2}-a} = 2(\pi)^{\frac{1}{2}} \quad (\tau \to \pm\infty);$$

therefore for some finite positive numbers C_1, C_2

$$C_1(1+|\tau|)^{a-\frac{1}{2}} < |\Gamma(a+ib+i\tau)|e^{\frac{1}{2}\pi|\tau|} < C_2(1+|\tau|)^{a-\frac{1}{2}} \quad (-\infty < \tau < \infty).$$

Hence there are two positive finite constants, c, C, depending on η and α only, such that

$$c(1+|\tau|)^{-\Re(\alpha)} < |\Gamma(\eta+1/p'-i\tau)/\Gamma(\eta+\alpha+1/p'-i\tau)| < C(1+|\tau|)^{-\Re(\alpha)} \tag{6.1}$$

for $-\infty < \tau < \infty$; hence from (4.3) we get

$$|f_1(\tau)| > C^{-1}|g_1(\tau)(1+|\tau|)^\alpha| > C^{-1}|g_1(\tau)\tau^\alpha|, \tag{6.2}$$

and, since $f_1(\tau) \in L_2(-\infty,\infty)$, the condition $\tau^\alpha g_1(\tau) \in L_2(-\infty,\infty)$ is necessary.

Conversely, when $g(z)$ belongs to $L_2(0,\infty)$ then $g_1(\tau)$ exists and belongs to $L_2(-\infty,\infty)$, and, since $\tau^\alpha g_1(\tau)$ also belongs to $L_2(-\infty,\infty)$, the function $f_1(\tau)$ defined by (4.3) satisfies the inequality

$$\int\limits_{-\infty}^{\infty} |f_1(\tau)|^2\,d\tau < c^{-2}\int\limits_{-\infty}^{\infty} |g_1(\tau)(1+|\tau|)^\alpha|^2\,d\tau$$

$$< K\int\limits_{-\infty}^{\infty} dt\,\{|g_1(\tau)|^2+|\tau^\alpha g_1(\tau)|^2\} < \infty;$$

therefore $f_1(\tau) \in L_2(-\infty, \infty)$, and the function $f(t) = \mathfrak{M}^{-1}f_1(\tau)$ exists and belongs to $L_2(0, \infty)$; evidently $I^+_{\eta,\alpha}f$ is equal to $g(z)$ almost everywhere.

The same argument gives 7 (*b*); the condition is scarcely sufficient, since probably the function $f_1(\tau)$ defined by (4.3) is not necessarily the Mellin transform of a function $f(t) \in L_p(0, \infty)$. Now let $2 < p \leqslant \infty$ and $g(z) \in \mathfrak{M}_p$, that is to say, $g(z)$ has a Mellin transform

$$g_1(\tau) \in L_{p'}(-\infty, \infty),$$

and let $\tau^\alpha g_1(\tau) \in L_{p'}(-\infty, \infty)$. Then plainly $f_1(\tau)$, defined by (4.3), belongs to $L_{p'}(-\infty, \infty)$ also, and, since $1 \leqslant p' < 2$, the function $f = \mathfrak{M}^{-1}f_1$ exists and belongs to $L_p(0, \infty)$,* we can easily see that $I^+_{\eta,\alpha}f = g$. Of course, the condition is not necessary, as we see by taking $g = I^+_{\eta,\alpha}\phi$, where ϕ belongs to L_p but has no Mellin transform. So Theorem 7 is proved.

Taking $\eta = 0$ or $\eta = -\alpha$ in $I^+_{\eta,\alpha}$ or $K^-_{\eta,\alpha}$ respectively, we obtain the

COROLLARY. *A necessary and sufficient condition that, given $\phi(z)$, the equation*

$$\phi(z) = \{\Gamma(\alpha)\}^{-1} \int\limits_0^z (z-t)^{\alpha-1} f(t)\, dt \quad (\Re(\alpha) > 0)$$

or
$$\phi(z) = \{\Gamma(\alpha)\}^{-1} \int\limits_z^\infty (t-z)^{\alpha-1} f(t)\, dt \quad (0 < \Re(\alpha) < \tfrac{1}{2})$$

should have a solution $f(t) \in L_2(0, \infty)$ is

$$z^{-\alpha}\phi(z) \in L_2(0, \infty) \quad \textit{and} \quad \tau^\alpha \mathfrak{M}\{z^{-\alpha}\phi(z)\} \in L_2(-\infty, \infty).$$

Let $0 < \Re(\alpha) < 1$ and, according to the definitions given by Hardy-Littlewood and Love-Young,

$$\phi^{\pm}_{-\alpha}(z) = \frac{d}{dz}\phi^{+}_{1-\alpha}(z),$$

then from (3.73) we easily have

$$f(z) \equiv \frac{d}{dz}\phi^{+}_{1-\alpha}(z) = \phi^{\pm}_{-\alpha}(z).$$

Hence the condition above is sufficient for $\phi^{\pm}_{-\alpha}(z)$ to exist and belong to $L_2(0, \infty)$.

We can easily extend this result to the case $0 < \Re(\alpha) < \infty$ using the more general definition of $\phi^{\pm}_{-\alpha}(z)$ given by Tamarkin.

* Cf. 4.2.

7. Representations of the operators by transforms of the Hankel type

Let $\Re(\nu) > -1$ and let $\psi = H_\nu \phi$ denote the Hankel transform

$$\psi(x) = \underset{N\to\infty}{\text{l.i.m. sq.}} \int\limits_0^N J_\nu(tx)(tx)^{\frac{1}{2}}\phi(t)\,dt \quad (\phi \in L_2(0,\infty)). \tag{7.1}$$

Here we take the Hankel transform in Tricomi's form $F(x) = \mathfrak{H}_\nu f$,

$$F(x) = \underset{N\to\infty}{\text{l.i.m. sq.}} \int\limits_0^N J_\nu\{2(tx)^{\frac{1}{2}}\}f(t)\,dt. \tag{7.2}$$

Putting $f(t) = (2t)^{-\frac{1}{4}}\phi\{(2t)^{\frac{1}{2}}\}$, $F(x) = (2x)^{-\frac{1}{4}}\psi\{(2x)^{\frac{1}{2}}\}$ and considering the equations $|f|_2 = |\phi|_2$, $|F|_2 = |\psi|_2$, from the theory of the Hankel transform we see that $f \in L_2(0,\infty)$ implies $F \in L_2(0,\infty)$, and from the theory of general transforms,* that (7.2) is equivalent to

$$F_1(\tau) = f_1(-\tau)\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}+i\tau)/\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}-i\tau) \quad (f_1 = \mathfrak{M}f;\ F_1 = \mathfrak{M}F). \tag{7.3}$$

We now split up (4.3) into two equations, taking

$$p = 2, \qquad \Re(\eta) > -\tfrac{1}{2}x.$$

$$F_1(\tau) = \frac{\Gamma(\eta+\frac{1}{2}+i\tau)}{\Gamma(\eta+\alpha+\frac{1}{2}-i\tau)} f_1(-\tau);\ g_1^+(\tau) = \frac{\Gamma(\eta+\alpha+\frac{1}{2}+i\tau)}{\Gamma(\eta+\alpha+\frac{1}{2}-i\tau)} F_1(-\tau).$$

Hence we get

$$F(x) = \underset{N\to\infty}{\text{l.i.m. sq.}} \int\limits_0^N J_{2\eta+\alpha}\{2(xt)^{\frac{1}{2}}\}(xt)^{-\frac{1}{2}\alpha}f(t)\,dt, \tag{7.4}$$

$$g^+_{\eta,\alpha} = \mathfrak{H}_{2\eta+2\alpha}F, \tag{7.5}$$

and

THEOREM 8. *The operator* $g^+_{\eta,\alpha} = I^+_{\eta,\alpha}f$ $(p = 2,\ \Re(\eta) > -\frac{1}{2})$ *is the product of the two operators defined by* (7.4) *and* (7.5).

* This immediately follows from the theory of 'general transforms', G. N. Watson, *Proc. London Math. Soc.* (2) 35 (1933), 156–99, cf. I. W. Busbridge, *J. of London Math. Soc.* 9 (1934), 179–87, and H. Kober, *Quart. J. of Math.* (Oxford) 8 (1937), 172–85. Under some hypotheses on $K(x)$, the equation

$$F(x) = \text{l.i.m. sq.} \int\limits_{1/N}^N K(xt)f(t)\,dt \quad (f \in L_2(0,\infty))$$

is equivalent to $F_1(\tau) = \omega(\tau)f_1(-\tau)$, when we define $\omega(\tau) = \lim\limits_{N\to\infty} \int\limits_{1/N}^N K(x)x^{-\frac{1}{2}+i\tau}\,dx$ (cf. Kober, § 3). The integral in (7.4) is absolutely convergent for $\Re(\alpha) > \frac{1}{2}$; the operator S was in substance dealt with by Kober, *Quart. J. of Math.* (Oxford) 9 (1938), 41–52, § 5.

Let $S_{2\eta+\alpha,\alpha}$ denote the operator (7.4); then we have

$$I^+_{\eta,\alpha} = \mathfrak{H}_{2\eta+2\alpha} S_{2\eta+\alpha,\alpha}. \tag{7.6}$$

By a similar argument and by (3.1) we get for $p = 2$

$$K^-_{\eta,\alpha} = S_{2\eta+\alpha,\alpha} \mathfrak{H}_{2\eta+2\alpha} \quad (\Re(\eta) > -\tfrac{1}{2}).$$

$$I^-_{\eta,\alpha} = S_{-2\eta-\alpha,\alpha} \mathfrak{H}_{-2\eta}; \qquad K^+_{\eta,\alpha} = \mathfrak{H}_{-2\eta} S_{-2\eta-\alpha,\alpha} \quad (\Re(\eta+\alpha) < \tfrac{1}{2}), \tag{7.7}$$

and we have $\quad K^-_{\eta,\alpha} I^+_{\eta,\alpha} = I^+_{\eta,\alpha} K^-_{\eta,\alpha} = S_{2\eta+\alpha,\alpha} S_{2\eta+\alpha,\alpha}$.

We also can get the decompositions

$$I^+_{\eta,\alpha} = S_{2\eta+\alpha,\alpha} \mathfrak{H}_{2\eta}, \qquad K^-_{\eta,\alpha} = \mathfrak{H}_{2\eta} S_{2\eta+\alpha,\alpha}$$

and the following generalization, due to Erdélyi, valid for

$$0 \leqslant \Re(\beta) \leqslant \Re(\alpha),$$

$$I^+_{\eta,\alpha} = S_{2\eta+\alpha+\beta,\alpha-\beta} S_{2\eta+\beta,\beta}; \qquad K^-_{\eta,\alpha} = S_{2\eta+\beta,\beta} S_{2\eta+\alpha+\beta,\alpha-\beta}. \tag{7.8}$$

when we put $S_{\nu,0} = \mathfrak{H}_\nu$. Taking $\eta = 0$ and $\eta = -a$, for instance, we have

$$\frac{1}{\Gamma(\alpha)} \int\limits_0^x f(t)(x-t)^{\alpha-1}\, dt = x^\alpha S_{\alpha+\beta,\alpha-\beta} S_{\beta,\beta} f \quad (0 \leqslant \Re(\beta) \leqslant \Re(\alpha)),$$

$$\frac{1}{\Gamma(\alpha)} \int\limits_x^\infty f(t)(t-x)^{\alpha-1}\, dt = x^\alpha S_{\beta-2\alpha,\beta} S_{\beta-\alpha,\alpha-\beta} f \quad (0 \leqslant \Re(\beta) \leqslant \Re(\alpha) < \tfrac{1}{2}),$$

when $p = 2$, $f \in L_2(0,\infty)$, $\Re(\alpha) > 0$.

SOME REMARKS ON HANKEL TRANSFORMS

By A. ERDÉLYI (*Edinburgh*) and H. KOBER (*Birmingham*)

[Received 7 May 1940]

1. The object of the present paper is to prove some theorems connecting Hankel transforms of different order.

As in the preceding paper by H. Kober,† we denote the Hankel transform of a function $f(x) \in L_p(0,\infty)$ by‡

$$F(x) = \mathfrak{H}_\nu f = \underset{N\to\infty}{\overset{\text{index } p'}{\text{l.i.m.}}} \int\limits_0^N J_{\nu,m}\{2\sqrt{(tx)}\}f(t)\,dt \quad (1 \leqslant p \leqslant 2), \tag{1}$$

where the limit in mean can be replaced by the ordinary integral over $(0,\infty)$ when $p = 1$, and we shall assume that $\frac{1}{2}\Re(\nu)+1/p' \neq 0$, -1, -2,.... Here $m = 0$ when $\frac{1}{2}\Re(\nu) > 1/p-1 = -1/p'$; otherwise m is the positive integer for which $1/p-1 < \frac{1}{2}\Re(\nu)+m < 1/p$. Also

$$J_{\nu,m}(z) = \sum_{n=m}^{\infty} \frac{(-)^n(\frac{1}{2}z)^{\nu+2n}}{n!\Gamma(\nu+n+1)}$$

$$= \frac{(-)^m(\frac{1}{2}z)^{\nu+2m}}{m!\Gamma(\nu+m+1)}\,{}_1F_2(1; m+1, \nu+m+1; -\tfrac{1}{4}z^2), \tag{2}$$

so that $J_{\nu,m}(z) = J_\nu(z)$ for $\frac{1}{2}\Re(\nu) > 1/p-1$. When $\frac{1}{2}\Re(\nu) < 1/p-1$, we shall call $\mathfrak{H}_\nu$ the cut Hankel transform. Though some fundamental properties of the cut Hankel transform are not different from those of the ordinary one,§ yet in some points ordinary and cut Hankel transforms differ from each other. Both ordinary and cut Hankel transforms together form what we call simply *Hankel transforms*.

The above choice of the non-negative integer m will be maintained throughout the paper.

The notation (1) of Hankel transform is slightly different from the notation generally used.‖ Our notation, which is identical with that

† This paper will be quoted as K so that K Theorem 2 or K (3.4) mean Theorem 2 or equation (3.4) of the preceding paper respectively.

‡ K (7.2).

§ H. Kober, *Quart. J. of Math.* (Oxford) 8 (1937), 186–99.

‖ Cf. e.g. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), 245 (9.1.3) and the literature quoted in this work, for the Hankel transform in the restricted sense. For the cut Hankel transform see H. Kober. loc. cit.

used by Tricomi,† has some advantages from the formal point of view. Its only disadvantage is that it does not reduce, for $\nu = \frac{1}{2}$ and $-\frac{1}{2}$ respectively, to Fourier's cosine or sine transform in the usual form.

The fundamental properties of $\mathfrak{H}_\nu$ are expressed in the following two theorems:

THEOREM 1. *Let* $1 \leqslant p \leqslant 2$ *and let* $f(t)$ *belong to* $L_p(0,\infty)$. *Then the transform* $\mathfrak{H}_\nu$ *as defined by* (1) *exists and*

$$\left\{\int_0^\infty |F(x)|^{p'}\,dx\right\}^{1/p'} \leqslant A\left\{\int_0^\infty |f(x)|^p\,dx\right\}^{1/p}.$$

When $p = 1$, *the left-hand term is to be replaced by the essential upper bound of* $|F(x)|$ *in* $(0,\infty)$.

THEOREM 2. *Let* $1 \leqslant p \leqslant 2$, *let* $f_j(t)$ *belong to* $L_p(0,\infty)$ *and* $F_j(x) = \mathfrak{H}_\nu f_j$ $(j = 1,2)$. *Then*

$$\int_0^\infty f_1 F_2\,dx = \int_0^\infty F_1 f_2\,dx.$$

This is Parseval's theorem.

These theorems (except for $p = 2$) are not trivially equivalent to the known fundamental theorems on Hankel transform in the usual notation. The proof of Theorems 1 and 2 is given in the Appendix to the present paper.

2. In the following two sections we shall give some theorems on the reduction of cut Hankel transform by repeated integrations by parts. These theorems have been the starting-point of the present investigations.

THEOREM 3. *Let* l *be a positive integer,* $\frac{1}{2}\Re(\nu)+1/p \neq 0, -1, -2,\ldots$ *and* $\frac{1}{2}\Re(\nu)+l < 1/p'$; *let* $f(t)$ *belong to* $L_p(0,\infty)$ $(1 \leqslant p \leqslant 2)$ *and*

(*a*) $F = \mathfrak{H}_\nu f$; *let*

$$g(z) = I^-_{\frac{1}{2}\nu,l} f = \frac{z^{-\frac{1}{2}\nu-l}}{\Gamma(l)} \int_z^\infty (t-z)^{l-1} f(t) t^{\frac{1}{2}\nu}\,dt \tag{3}$$

and $G = I^-_{\frac{1}{2}\nu,l} F$; *then* $G = \mathfrak{H}_{\nu+2l}\, g$.

† F. Tricomi, *Rend. dei Lincei* (6) 22 (1935), 564–71 and 572–6, *Atti della Reale Acc. d. Sci. di Torino* 71 (1935), 1–7.

(b) $F = \mathfrak{H}_{\nu+2l}f$; *let*

$$g(z) = K^{+}_{\frac{1}{2}\nu,l}f = \frac{z^{\frac{1}{2}\nu}}{\Gamma(l)}\int\limits_0^z (z-t)^{l-1}f(t)t^{-\frac{1}{2}\nu-l}\,dt \tag{4}$$

and $G = K^{+}_{\frac{1}{2}\nu,l}F$; *then* $G = \mathfrak{H}_{\nu}g$.

We shall prove these theorems for $l = 1$ only, The more general statement then immediately follows by repeated application of that particular case.

Proof of Theorem 3 (*a*) *for* $l = 1$. By Theorem 1, $F(x) \in L_{p'}(0,\infty)$. Also, since $\frac{1}{2}\Re(\nu)+1 < 1/p' \leqslant 1/p$, $g(x)$ and $G(x)$ belong to $L_p(0,\infty)$ and $L_{p'}(0,\infty)$ respectively.†

Now we define

$$f_1(t) = 0 \quad \text{when} \quad 0 \leqslant t < x \quad \text{and} \quad f_1(t) = t^{\frac{1}{2}\nu} \quad \text{when} \quad t > x.$$

Plainly, $f_1(t) \in L_p(0,\infty)$ for every fixed $x > 0$, and from (2) it is easily seen that

$$F_1(t) = \mathfrak{H}_{\nu}f_1 = \int\limits_x^\infty J_{\nu,m}\{2\sqrt{(tu)}\}u^{\frac{1}{2}\nu}\,du = -x^{\frac{1}{2}\nu+\frac{1}{2}}t^{-\frac{1}{2}}J_{\nu+1,m}\{2\sqrt{(tx)}\}. \tag{5}$$

The application of Theorem 2 to $f(t)$, $F(t)$ and $f_1(t)$, $F_1(t)$ gives

$$\int\limits_x^\infty F(t)t^{\frac{1}{2}\nu}\,dt = -x^{\frac{1}{2}\nu+\frac{1}{2}}\int\limits_0^\infty J_{\nu+1,m}\{2\sqrt{(xt)}\}f(t)t^{-\frac{1}{2}}\,dt,$$

i.e.
$$G(x) = x^{-\frac{1}{2}\nu-1}\int\limits_x^\infty F(t)t^{\frac{1}{2}\nu}\,dt = -\int\limits_0^\infty J_{\nu+1,m}\{2\sqrt{(xt)}\}f(t)(xt)^{-\frac{1}{2}}\,dt$$

$$= x^{-\frac{1}{2}}\int\limits_0^\infty J_{\nu+1,m}\{2\sqrt{(xt)}\}t^{-\frac{1}{2}\nu-\frac{1}{2}}d\{t^{\frac{1}{2}\nu+1}g(t)\}.$$

Hence $G(x) = \lim\limits_{N\to\infty} G(x,N)$, where

$$G(x,N) = x^{-\frac{1}{2}}\int\limits_0^N J_{\nu+1,m}\{2\sqrt{(xt)}\}t^{-\frac{1}{2}\nu-\frac{1}{2}}\,d\{t^{\frac{1}{2}\nu+1}g(t)\}. \tag{6}$$

† For $1 \leqslant p < \infty$ this follows from Theorem 330 of Hardy-Littlewood-Pólya's *Inequalities* (Cambridge, 1934), the case $p = 1$ being covered by the note to this theorem. When $p = 1$, we have $p' = \infty$, $|G|_{p'} =$ essential upper bound of $|G(x)|$, and

$$|G(z)| \leqslant z^{-\frac{1}{2}\Re(\nu)-1}\int\limits_z^\infty t^{\frac{1}{2}\Re(\nu)}|g(t)|\,dt \leqslant |\tfrac{1}{2}\Re(\nu)+1|^{-1}|g|_{p'},$$

and so $G \in L_{p'} = L_\infty$.

Now, integrating by parts, we have

$$G(x,N) = x^{-\frac{1}{2}}[J_{\nu+1,m}\{2\sqrt{(xt)}\}t^{\frac{1}{2}}g(t)]_{t=0}^{N} -$$

$$-x^{-\frac{1}{2}}\int_0^N \frac{d}{dt}[t^{-\frac{1}{2}\nu-\frac{1}{2}}J_{\nu+1,m}\{2\sqrt{(xt)}\}]g(t)t^{\frac{1}{2}\nu+1}\,dt$$

$$= G_1(x,N)+G_2(x,N). \tag{7}$$

Clearly, we have from (2) for $t \to 0$

$$t^{\frac{1}{2}}J_{\nu+1,m}\{2\sqrt{(xt)}\} = O(t^{\frac{1}{2}\nu+m+1}) \quad \text{with} \quad \tfrac{1}{2}\Re(\nu)+m+1 > 1/p$$

and, from (2) and the well-known asymptotic representations of Bessel functions of the first kind,† for $t \to \infty$

$$t^{\frac{1}{2}}J_{\nu+1,m}\{2\sqrt{(xt)}\} = O(t^{\frac{1}{4}})+O(t^{\frac{1}{2}\nu+m}).$$

Also we have from Hölder's inequality and (3) for all values of t

$$|g(t)| \leqslant t^{-\frac{1}{2}\Re(\nu)-1}\left\{\int_0^\infty |f(t)|^p\,dt\right\}^{1/p}\left\{\int_t^\infty u^{\frac{1}{2}\Re(\nu)p'}\,du\right\}^{1/p'} < At^{-1/p} \quad (p>1),$$

$$|g(t)| \leqslant t^{-\frac{1}{2}\Re(\nu)-1}\int_t^\infty |f(\tau)|\tau^{\frac{1}{2}\Re(\nu)}\,d\tau \leqslant t^{-1}\int_t^\infty |f(\tau)|\,d\tau \leqslant At^{-1} \quad (p=1),$$

where A does not depend on t. Hence the expression in the square brackets in $G_1(x,N)$ vanishes for $t=0$, and also

$$G_1(x,N) = O(N^{\frac{1}{4}-1/p})+O(N^{\frac{1}{2}\nu+m-1/p}) \to 0 \tag{8}$$

for $N\to\infty$ under our hypotheses on p, m, ν.

Again, we have from (2),

$$\frac{d}{dt}[t^{-\frac{1}{2}\nu-\frac{1}{2}}J_{\nu+1,m}\{2\sqrt{(xt)}\}] = -x^{\frac{1}{2}}t^{-\frac{1}{2}\nu-1}J_{\nu+2,m-1}\{2\sqrt{(xt)}\}, \tag{9}$$

and hence $$G_2(x,N) = \int_0^N J_{\nu+2,m-1}\{2\sqrt{(xt)}\}g(t)\,dt, \tag{7'}$$

and $$G(x) = \lim_{N\to\infty} G_2(x,N)$$

by (7) and (8). Now $g(x) \in L_p$; therefore by Theorem 1 and (7′) the limit in mean of index p' of $G_2(x,N)$ exists and equals $\mathfrak{H}_{\nu+2}g$. Hence, by a well-known theorem, $G(x) = \mathfrak{H}_{\nu+2}g$. This completes the proof.

The proof of Theorem 3 (b) is rather similar to that of 3(a), except that the integration by parts has to be done the other way round, as it were.

† Cf. also H. Kober, *Quart. J. of Math.*, loc. cit. (2.3).

Instead of the function f_1, we have to use here

$$f_2(t) = t^{-\frac{1}{2}\nu-1} \quad \text{when} \quad 0 \leqslant t < x \quad \text{and} \quad f_2(t) = 0 \quad \text{when} \quad t > x.$$

Instead of (5) we obtain

$$F_2(t) = \mathfrak{H}_{\nu+2} f_2 = -t^{-\frac{1}{2}} x^{-\frac{1}{2}\nu-\frac{1}{2}} J_{\nu+1,m}\{2\surd(tx)\}; \tag{10}$$

we then apply Theorem 2 to f, f_2 and F, F_2 writing in this case

$$f(t) = t^{\frac{1}{2}\nu+1} \frac{d}{dt}\{g(t)t^{-\frac{1}{2}\nu}\} \tag{11}$$

and integrating by parts. In doing so, instead of (9) we have to use

$$\frac{d}{dt}[t^{\frac{1}{2}\nu+\frac{1}{2}} J_{\nu+1,m}\{2\surd(tx)\}] = x^{\frac{1}{2}} t^{\frac{1}{2}\nu} J_{\nu,m}\{2\surd(tx)\}.$$

The rest of the proof follows closely that of Theorem 3 (a).

3. The inversion of Theorem 3 is given by the following.

THEOREM 4. *Let l be a positive integer, $\frac{1}{2}\Re(\nu)+1/p \neq 0, -1, -2,\ldots$ and $\frac{1}{2}\Re(\nu)+l < 1-1/p$; let $g(t)$ belong to $L_p(0,\infty)$ $(1 \leqslant p \leqslant 2)$ and*

(a) *$G = \mathfrak{H}_{\nu+2l}\, g$. Suppose that g is representable in the form $g = I^-_{\frac{1}{2}\nu,l} f$ where $f(t) \in L_p(0,\infty)$. Then also G is representable in the form $G = I^-_{\frac{1}{2}\nu,l} F$, and $F = \mathfrak{H}_\nu f$.*

(b) *$G = \mathfrak{H}_\nu\, g$. Suppose that g is representable in the form $g = K^+_{\frac{1}{2}\nu,l} f$ where $f(t) \in L_p(0,\infty)$. Then also G is representable in the form $G = K^+_{\frac{1}{2}\nu,l} F$, and $F = \mathfrak{H}_{\nu+2l} f$.*

Again, it is sufficient to prove Theorem 4 (a) only. From Theorem 1, $F = \mathfrak{H}_\nu f$ exists and belongs to $L_{p'}(0,\infty)$. Also $G^* = I^-_{\frac{1}{2}\nu,l} F$ is equal to $\mathfrak{H}_{\nu+2l}\, g$, by Theorem 3 (a), and hence almost everywhere equal to G. Thus $G = I^-_{\frac{1}{2}\nu,l} F$ almost everywhere, and the theorem is proved.

When $p = 2$, we can use the last two theorems to reduce any cut Hankel transform to the ordinary one. For $1 \leqslant p < 2$ this is not the case when $1/p' < \frac{1}{2}\Re(\nu)+m < 1/p$. Hence the class $L_2(0,\infty)$ plays a particular role, and in §§ 5–7 we shall restrict ourselves to functions of this class.

The method used here is not confined to cut Hankel transforms nor is the restriction imposed upon l in our theorems unavoidable. Using appropriate limits of integration, we can start, at least in $L_2(0,\infty)$, with any Hankel transform, ordinary or cut, and integrate by parts in either way any number of times. From the results of §§ 5, 6 it will be seen that even fractional integration by parts may be used.

4. It is of some consequence for the theory of biorthogonal systems that the reduction of $\mathfrak{H}_\nu$ to $\mathfrak{H}_{\nu+2l}$ is twofold. Let both $\{f_n\}$ and $\{\phi_n\}$ ($n = 0, 1, 2,...$) be sequences of eigen-functions of $\mathfrak{H}_{\nu+2l}$; furthermore, let $\{f_n \phi_n\}$ be a biorthogonal system. Then we can show that the sequences

$$g_n = (I^-_{\frac{1}{2}\nu,l})^{-1} f_n, \qquad \psi_n = K^+_{\frac{1}{2}\nu,l}\phi_n \quad (n = 0, 1, 2,...)$$

are both systems of eigen-functions of $\mathfrak{H}_\nu$ and also that $\{g_n, \psi_n\}$ is a biorthogonal system. This procedure has been used actually to obtain biorthogonal sets of eigen-functions of the cut Hankel transform from those of the ordinary one.†

5. Theorems 3 and 4 are particular cases of some more general rules indicated at the end of § 3. We shall, however, confine ourselves in the following sections to the case $p = 2$ and shall give only three more theorems. Also we shall employ another method of proof using the theory of 'general transforms' and the results of the preceding paper, since this procedure is extremely short and gives better results than that sketched above.

Throughout the rest of this paper α is an arbitrary parameter such that $\Re(\alpha) \geqslant 0$. For the definitions and fundamental properties of the operators $I^{\pm}_{\eta,\alpha}$ and $K^{\pm}_{\eta,\alpha}$ we refer the reader to K.‡

In this section we state those of our theorems which refer to Hankel transform in the restricted sense only.

THEOREM 5. *Let* $\Re(\nu) > -1$, $f(t) \in L_2(0,\infty)$ *and*

(*a*) $F = \mathfrak{H}_\nu f$; *let* $g = I^+_{\frac{1}{2}\nu,\alpha} f$ *and* $G = I^+_{\frac{1}{2}\nu,\alpha}$; *then* $G = \mathfrak{H}_{\nu+2\alpha}\, g$.

(*b*) $F = \mathfrak{H}_{\nu+2\alpha} f$; *let* $g = K^-_{\frac{1}{2}\nu,\alpha} f$ *and* $G = K^-_{\frac{1}{2}\nu,\alpha} F$; *then* $G = \mathfrak{H}_\nu\, g$.

Proof of Theorem 5 (*a*). In consequence of the theory of Hankel transforms§ F exists and belongs to $L_2(0,\infty)$. Also||

$$\frac{F_1(\tau)}{f_1(-\tau)} = \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}+i\tau)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}-i\tau)}.$$

† A. Erdélyi, *Proc. Edinburgh Math. Soc.* (In the press.)

‡ In K $\Re(\alpha) > 0$ has been assumed. In another paper by Kober it will be shown that the operators can be defined for $\Re(\alpha) = 0$ and that the theorems we shall make use of hold for $\Re(\alpha) \geqslant 0$ when $p = 2$.

§ E. C. Titchmarsh, *Proc. Cambridge Phil. Soc.* 21 (1923), 463–73.

|| $\Phi_1(\tau)$ denotes the Mellin transform of $\Phi(z) \in L_2(0,\infty)$, $\Phi_1(\tau) = \mathfrak{M}\Phi$.

From K Theorem 2 we see that g and G exist and belong to $L_2(0,\infty)$. Furthermore, in consequence of K Theorem 5 (*a*),

$$\frac{G_1(\tau)}{F_1(\tau)} = \frac{g_1(\tau)}{f_1(\tau)} = \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}-i\tau)}{\Gamma(\frac{1}{2}\nu+\alpha+\frac{1}{2}-i\tau)}. \tag{12}$$

Thus $$\frac{G_1(\tau)}{g_1(-\tau)} = \frac{G_1(\tau)}{F_1(\tau)}\frac{F_1(\tau)}{f_1(-\tau)}\frac{f_1(-\tau)}{g_1(-\tau)} = \frac{\Gamma(\frac{1}{2}\nu+\alpha+\frac{1}{2}+i\tau)}{\Gamma(\frac{1}{2}\nu+\alpha+\frac{1}{2}-i\tau)},$$

and hence in consequence of the theory of Hankel transform we have $G = \mathfrak{H}_{\nu+2\alpha}\, g$.

We obtain an alternative proof of Theorem 5 (*a*) from the results of K § 7 using the two alternative decompositions of $I^+_{\frac{1}{2}\nu,\alpha}$. The statement of Theorem 5 (*a*) reads

$$I^+_{\frac{1}{2}\nu,\alpha}\mathfrak{H}_\nu = \mathfrak{H}_{\nu+2l}\, I^+_{\frac{1}{2}\nu,\alpha}. \tag{13}$$

Now from K (7.6) we have, since $S_{\nu,0} = \mathfrak{H}_\nu$:

$$I^+_{\frac{1}{2}\nu,\alpha}\mathfrak{H}_\nu = S_{\nu+2\alpha,0}\, S_{\nu+\alpha,\alpha}\, S_{\nu,0},$$

and from K (7.7)

$$\mathfrak{H}_{\nu+2\alpha}\, I^+_{\frac{1}{2}\nu,\alpha} = S_{\nu+2\alpha,0}\, S_{\nu+\alpha,\alpha}\, S_{\nu,0}.$$

Hence (13) is true.

The proof of Theorem 5 (*b*) is quite similar.

THEOREM 6. *Let* $\Re(\nu) > -1$, $g(t) \in L_2(0,\infty)$ *and*

$$\tau^\alpha g_1(\tau) = \tau^\alpha \mathfrak{M} g(t) \in L_2(-\infty,\infty);$$

let furthermore

(*a*) $G = \mathfrak{H}_{\nu+2\alpha}\, g$. *Then* $\tau^\alpha G_1(\tau) = \tau^\alpha \mathfrak{M} G(t) \in L_2(-\infty,\infty)$, *the functions* $f = (I^+_{\frac{1}{2}\nu,\alpha})^{-1} g$ *and* $F = (I^+_{\frac{1}{2}\nu,\alpha})^{-1} G$ *exist, belong to* $L_2(0,\infty)$, *and* $F = \mathfrak{H}_\nu f$.

(*b*) $G = \mathfrak{H}_\nu\, g$. *Then* $\tau^\alpha G_1(\tau) = \tau^\alpha \mathfrak{M} G(t) \in L_2(-\infty,\infty)$, *the functions* $f = (K^-_{\frac{1}{2}\nu,\alpha})^{-1} g$ *and* $F = (K^-_{\frac{1}{2}\nu,\alpha})^{-1} G$ *exist, belong to* $L_2(0,\infty)$, *and* $F = \mathfrak{H}_{\nu+2\alpha} f$.

Proof of Theorem 6 (*a*). By the theory of Hankel transform $G(x)$ exists, belongs to $L_2(0,\infty)$; its Mellin transform exists and

$$|\tau^\alpha G_1(\tau)| = \left|\tau^\alpha \frac{\Gamma(\frac{1}{2}\nu+\alpha+\frac{1}{2}+i\tau)}{\Gamma(\frac{1}{2}\nu+\alpha+\frac{1}{2}-i\tau)} g_1(-\tau)\right| < A\,|\tau^\alpha g_1(-\tau)|$$

where A is independent of τ. Hence $\tau^\alpha g_1(\tau) \in L_2(-\infty,\infty)$ implies $\tau^\alpha G_1(\tau) \in L_2(-\infty,\infty)$, and by K Theorem 7 (*a*) both of the functions $f = (I^+_{\frac{1}{2}\nu,\alpha})^{-1} g$ and $F = (I^+_{\frac{1}{2}\nu,\alpha})^{-1} G$ exist and belong to $L_2(0,\infty)$. Consequently, $\Phi = \mathfrak{H}_\nu f$ exists and belongs to $L_2(0,\infty)$; also $\Psi = I^+_{\frac{1}{2}\nu,\alpha}\Phi$

exists, belongs to $L_2(0,\infty)$ and from Theorem 5 (*a*) we have $\Psi = \mathfrak{H}_{\nu+2\alpha} g$. But according to the premisses of Theorem 6 (*a*), $G = \mathfrak{H}_{\nu+2\alpha} g$. Hence $\Psi \equiv G$ and, the operation $(I^{+}_{\frac{1}{2}\nu,\alpha})^{-1}$ being unique (K § 5), also $\Phi \equiv g$. This completes the proof of $g = \mathfrak{H}_\nu f$. The proof of Theorem 6 (*b*) is similar.

Plainly Theorem 6 is the inversion of Theorem 5.

Theorems 5 and 6 can be extended to negative values of $\Re(\nu)+1$ when using appropriate generalizations of the operators $I^{+}_{\frac{1}{2}\nu,\alpha}$ and $K^{-}_{\frac{1}{2}\nu,\alpha}$. This will be dealt with in a subsequent paper.

6. This section contains a theorem which, in a certain sense, is a more precise statement of Theorem 4 in the particular case $p = 2$.

THEOREM 7. *Let l be a positive integer, $\Re(\nu) \neq -3, -5, -7,...$ and $\Re(\nu)+2l < 1$; let $g(t) \in L_2(0,\infty)$ and*

$$\tau^l g_1(\tau) = \tau^l \mathfrak{M} g(t) \in L_2(-\infty,\infty);$$

let furthermore

(*a*) $G = \mathfrak{H}_{\nu+2l} g$. *Then $\tau^l G_1(\tau) = \tau^l \mathfrak{M} G(t) \in L_2(-\infty,\infty)$, the functions $f = (I^{-}_{\frac{1}{2}\nu,l})^{-1} g$ and $F = (I^{-}_{\frac{1}{2}\nu,l})^{-1} G$ exist, belong to $L_2(0,\infty)$ and $F = \mathfrak{H}_\nu f$.*

(*b*) $G = \mathfrak{H}_\nu g$. *Then $\tau^l G_1(\tau) = \tau^l \mathfrak{M} G(t) \in L_2(-\infty,\infty)$, the functions $f = (K^{+}_{\frac{1}{2}\nu,l})^{-1} g$ and $F = (K^{+}_{\frac{1}{2}\nu,l})^{-1} G$ exist, belong to $L_2(0,\infty)$ and $F = \mathfrak{H}_{\nu+2l} f$.*

The proof of Theorem 7 is similar to that of Theorem 6.

7. Theorems 5–7 can also be obtained as particular cases of some well-known rules† connecting different classes of self-reciprocal functions; though we have to replace Titchmarsh's rule 3 by a slightly more general one‡ which also covers the case of the cut Hankel transform. Also some care is needed. Of course, if T is *any* transformation in $L_2(0,\infty)$ with domain $L_2(0,\infty)$ which transforms $\pm\mathfrak{H}_\mu$ into $\pm\mathfrak{H}_\nu$, i.e. such that $T(\pm\mathfrak{H}_\mu f) = \pm\mathfrak{H}_\nu Tf$ for any $f \in L_2(0,\infty)$, obviously T also transforms any function self or skew reciprocal with respect to $\mathfrak{H}_\mu$ into a function with the same property but with respect

† E. G. Phillips, *J. of London Math. Soc.* 4 (1929), 310–13; G. H. Hardy and E. C. Titchmarsh, *Proc. London Math. Soc.* (2) 33 (1931), 225–32; B. M. Mehrotra, *Proc. London Math. Soc.* (2) 34 (1932), 231–40, and *Proc. Edinburgh Math. Soc.* (2) 4 (1934), 53–6. Cf. also Titchmarsh's *Theory of Fourier Integrals* (Oxford 1937) §§ 9.14–9.16.

‡ H. Kober, *Proc. London Math. Soc.* (2) 45 (1939), 229–42, vide (3.2).

to $\mathfrak{H}_\nu$. Here we have to employ the converse, which is also true, if we add the condition that T is linear.†

8. Appendix

The proof of Theorems 1 and 2 follows the lines of the proofs given by one of the authors and by I. W. Busbridge for the Hankel transform in its customary form.‡ It is based upon the lemmas:

LEMMA 1. *Let ν be restricted and m be defined as in § 1. Then*

$$J_{\nu,m}(2\sqrt{x}) = \alpha_1 K_1(x)+\alpha_2 K_2(x)+K_3(x),$$

where $$\left.\begin{matrix}\alpha_1\\ \alpha_2\end{matrix}\right\} = \frac{1}{\sqrt{\pi}}\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi), \qquad K_2(x) = x^{-\frac{1}{4}}\sin(2\sqrt{x}),$$

while $$K_1(x) = \begin{cases}0,\\ x^{-\frac{1}{4}}\cos(2\sqrt{x})\end{cases}$$

and $$|K_3(x)| < \begin{cases}A(x^{\frac{1}{2}\Re(\nu)+m}+x^{\frac{1}{4}}), \\ A(x^{\frac{1}{2}\Re(\nu)+m-1}+x^{-\frac{3}{4}})\end{cases} \quad \textit{for} \quad \begin{cases}0<x<1,\\ x>1.\end{cases}$$

LEMMA 2. *Let N be any positive number, let $K_j(x)$ $(j = 1, 2)$ be defined as in Lemma 1, and let*

$$g_j(x) = T_j f = \int\limits_0^N K_j(tx)f(t)\,dt \quad (j = 1, 2).$$

Then

$$|g_1(x)|_2 \leqslant C\Big(\int\limits_0^N |f(t)|^2\,dt\Big)^{\frac{1}{2}}, \tag{14}$$

$$|g_2(x)|_2 \leqslant C\Big(\int\limits_0^N |f(t)|^2\,dt\Big)^{\frac{1}{2}}, \tag{15}$$

where C is an absolute constant, and

$$|g_j(x)|_\infty = \operatorname*{ess.\,u.b.}_{0<x<\infty} |g_j(x)| \leqslant C\int\limits_0^N |f(t)|\,dt \quad (j = 1, 2). \tag{16}$$

LEMMA 3. *Let $g_j(x)$ $(j = 1, 2)$ be defined as in Lemma 2. Then, for $1 \leqslant p \leqslant 2$,*

$$|g_j(x)|_{p'} \leqslant C\Big(\int\limits_0^N |f(t)|^p\,dt\Big)^{1/p} \quad (j = 1, 2).$$

† H. Kober, *Proc. London Math. Soc.* (2) 45 (1939), p. 239, § 2.

‡ H. Kober, referred to in § 1; I. W. Busbridge, *Quart. J. of Math.* (Oxford), 9 (1938), 148–60, Lemmas 1–3 and Theorems 2–4. For the notation of the norm of $f(t)$ in $L_p(0, \infty)$ by $|f|_p$ vide K § 1.

LEMMA 4. *Let $K_3(x)$ be defined as in Lemma 1, let $1 \leqslant p \leqslant 2$, and let*

$$g_3(x) = \int_0^\infty K_3(tx)f(t)\,dt.$$

Then $$|g_3(x)|_{p'} \leqslant A\,|f(t)|_p.$$

The proof of Lemma 1 easily follows from the asymptotic expansion of $J_\nu(z)$, that of (15) from a well-known property of the Fourier sine transform. Putting

$$\chi(x) = \int_0^x K_1(\xi)\,d\xi,$$

$$\omega(\tau) = (\tfrac{1}{2}-i\tau)\mathfrak{M}\{\chi(x)/x\} = \sqrt{\pi}\frac{\Gamma(\frac{1}{4}+i\tau)}{\Gamma(\frac{1}{4}-i\tau)} - 2^{-2i\tau+\frac{1}{2}}\int_0^2 x^{2i\tau-\frac{1}{2}}\cos x\,dx,$$

then $\omega(\tau)$ is bounded in $(-\infty,\infty)$, and so (14) follows from the theory of general transforms.† (16) is trivial, since $|K_j(x)| \leqslant 2$ $(j = 1,2)$. Lemma 3 follows from Lemma 2 by the convexity theorem of M. Riesz.‡ For Lemma 4 $(1 < p \leqslant 2)$ cf. Kober, loc. cit. 192, and Busbridge, loc. cit. 157; the lemma holds also when $p = 1$.

The proof of Theorem 1 is an easy consequence of Lemmas 3 and 4. The proof of Theorem 2 follows from

LEMMA 5. *Let $1 \leqslant p \leqslant \infty$; let $K(t,x)$ be a symmetric function of t and x in $(0,\infty;\,0,\infty)$ and bounded in any finite square $(1/a, a;\ 1/a, a)$, and let the functions*

$$g_j(x) = \underset{N\to\infty}{\overset{\text{index } p'}{\text{l.i.m.}}} \int_{1/N}^N K(t,x)f_j(t)\,dt \quad (f_j \in L_p(0,\infty),\ j = 1,2)$$

exist almost everywhere in $(0,\infty)$. Then

$$\int_0^\infty f_1(x)g_2(x)\,dx = \int_0^\infty f_2(t)g_1(t)\,dt.$$

† G. N. Watson, *Proc. London Math. Soc.* (2), 35 (1933), 156–99. H. Kober, *Quart. J. of Math.* (Oxford) 8 (1937), 172–85, Satz 2A.

‡ M. Riesz, *Acta Math.* 49 (1927), 465–97.

ON CERTAIN EXPANSIONS INVOLVING PRODUCTS OF LEGENDRE FUNCTIONS

By T. G. COWLING (*Manchester*)

[Received 2 July 1940]

IN considering the expression of the homogeneous solutions of the equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+k^2\frac{\partial^2\phi}{\partial z^2}=0$$

in terms of the similar solutions of Laplace's equation, I was led to the relations

$$(\mu^2+\mu'^2-1)^{\frac{1}{2}n}P_n^m\left\{\frac{\mu\mu'}{(\mu^2+\mu'^2-1)^{\frac{1}{2}}}\right\}$$
$$=\frac{(n+m)!}{n!}\sum_{r=0}^{n}\frac{(r-m)!}{(r+m)!}a_{nr}\,P_r^m(\mu)P_r^m(\mu'), \qquad (1)$$

$$(\mu^2+\mu'^2-1)^{-\frac{1}{2}(n+1)}P_n^m\left\{\frac{\mu\mu'}{(\mu^2+\mu'^2-1)^{\frac{1}{2}}}\right\}$$
$$=\frac{(-1)^m n!}{(n-m)!}\sum_{r=n}^{\infty}\frac{(r-m)!}{(r+m)!}b_{nr}\,P_r^m(\mu)Q_r^m(\mu'). \qquad (2)$$

In these, m, n are positive integers, and $0\leqslant m\leqslant n$; a_{nr} is the coefficient of $P_r(z)$ in the expansion of z^n in terms of Legendre polynomials, and b_{nr} that of $Q_r(z)$ in the similar expansion of z^{-n-1} in terms of Legendre functions of the second type,* so that

$$a_{nr}=0=b_{rn}$$

* The expansion of z^{-n-1} in terms of $Q_r(z)$ functions does not seem to be well known. It is derived by using C. Neumann's general expansion formula. If $f(z)$ is any function regular in the annulus between confocal ellipses α, β with $z=\pm 1$ as foci, of which α surrounds β, then throughout the annulus

$$f(z)=\sum_{r=0}^{\infty}c_r P_r(z)+\sum_{r=0}^{\infty}d_r Q_r(z),$$

where

$$c_r=\frac{(2r+1)}{2\pi\iota}\int_{\alpha}f(z)Q_r(z)\,dz,$$

$$d_r=\frac{(2r+1)}{2\pi\iota}\int_{\beta}f(z)P_r(z)\,dz.$$

If $f(z)=z^{-n-1}$, the dimensions of α can be taken indefinitely large, when it is clear that $c_r=0$. Also, by Cauchy's theorem, d_r is $(2r+1)$ times the coefficient of z^n in the expansion of P_r. The expansion is uniformly convergent save on the real axis between $z=\pm 1$.

if $r > n$ or $n-r$ is a positive odd integer; if $n-r$ is a positive even integer or zero, then

$$a_{nr} = (2r+1)\frac{n(n-1)...(n-r+2)}{(n+r+1)(n+r-1)...(n-r+3)},$$

$$b_{rn} = (2n+1)\frac{(-1)^{\frac{1}{2}(n-r)}(n+r)!}{2^n r!\{\frac{1}{2}(n+r)\}!\{\frac{1}{2}(n-r)\}!}.$$

Hobson's associated Legendre functions are used. The quantities μ, μ' are given in terms of x, y, z, k by

$$\mu = z/\surd\{x^2+y^2+z^2\}, \qquad \mu' = k/\surd(k^2-1).$$

Proof of (1). By the Laplacian integral for P_n^m, the expression on the left of (1) is equal to

$$\frac{(n+m)!}{n!\,\pi}\int_0^\pi \{\mu\mu'+\cos\phi\,\surd(\mu^2-1)\surd(\mu'^2-1)\}^n \cos m\phi\, d\phi$$

$$= \frac{(n+m)!}{n!\,\pi}\sum_{r=0}^{n} a_{nr}\int_0^\pi P_r\{\mu\mu'+\cos\phi\surd(\mu^2-1)\surd(\mu'^2-1)\}\cos m\phi\, d\phi,$$

and, since, by the addition theorem, the integral on the right is equal to

$$(r-m)!\,P_r^m(\mu)P_r^m(\mu')/(r+m)!,$$

the result follows.

Proof of (2). Let μ, μ' be positive, and such that $\mu < 1$, $\mu' > 1/\mu$. Then the expression on the left equals

$$\frac{n!}{(n-m)!\,\pi}\int_0^\pi \{\mu\mu'+\cos\phi\surd(\mu^2-1)\surd(\mu'^2-1)\}^{-n-1}\cos m\phi\, d\phi$$

$$= \frac{n!}{(n-m)!\,\pi}\sum_{r=n}^{\infty} b_{nr}\int_0^\pi Q_r\{\mu\mu'+\cos\phi\surd(\mu^2-1)\surd(\mu'^2-1)\}\cos m\phi\, d\phi,$$

whence, by use of the addition formula

$$Q_r\{\mu\mu'+\cos\phi\surd(\mu^2-1)\surd(\mu'^2-1)\}$$
$$= P_r(\mu)Q_r(\mu')+2\sum_{m=1}^{\infty}(-1)^m P_r^{-m}(\mu)Q_r^m(\mu')\cos m\phi,$$

together with the relation

$$P_r^{-m}(\mu) = \frac{(r-m)!}{(r+m)!}P_r^m(\mu) \quad (m \leqslant r),$$

the result follows. The assumptions limiting μ and μ' were made to ensure the validity of the addition formula; but, using the asymptotic expressions for $P_r^m(\mu)$, $Q_r^m(\mu')$ for large r, we find that each side of the equation is an analytic function of μ and μ' if μ lies inside an ellipse with foci ± 1, and μ' outside a larger ellipse with the same foci. Thus the relation is valid subject to these less stringent conditions.